Elastic Waves in a Homogeneous/ an Inhomogeneous Fibre-Reinforced Elastic Half-Spaces

C. Zorammuana1 and S. S. Singh2*
1Department of Education, Assam University, Silchar–788011, Assam, India
2Department of Mathematics & Computer Science, Mizoram University, Aizawl–796004, Mizoram, India
E-mail: *saratcha32@yahoo.co.uk

Abstract—The problem of reflection and refraction of elastic waves due to incident quasi P and quasi SV-waves at a plane interface between the homogeneous and inhomogeneous fibre-reinforced elastic half-spaces has been investigated. The reflection and refraction coefficients and energy ratios of the reflected and refracted quasi P and quasi SV-waves are obtained. These coefficients and ratios are computed numerically for a particular model and the results are shown graphically.

Keywords: Fibre-reinforced Elastic Half-space, qP and qSV-waves, Reflection and Refraction Coefficients, Energy Ratio

2010 Mathematics subject classification 2010: 74B15; 74J05; 78A40

INTRODUCTION

Spencer (1972) gave the concept of deformation in fibre-reinforced elastic materials. The concrete and steel components of fibre-reinforced materials act together as a single anisotropic unit as long as they remain in the elastic condition. Such composite materials are used in many engineering structures due to high strength and lightweight. The stress in elastic plates is reinforced by fibers lying in concentric circles (Belfield et al. 1983) with anisotropic characters. The reflection of quasi P and quasi SV-waves at a plane free and rigid boundaries of a fibre-reinforced elastic medium were investigated (Chattopadhyay et al. 2002) and the authors obtained the phase velocity of quasi P and quasi SV-waves. Singh and Singh (2004) discussed the reflection of plane waves at the free surface of a fibre-reinforced elastic half-space and derived the closed form expressions of the amplitude ratios for reflected qSV and qP-waves. The scattering of elastic waves from a corrugated interface between two dissimilar fibre-reinforced elastic half-spaces were discussed by Singh and Tomar (2007a) and they obtained the reflection and transmission coefficients with the help of the Rayleigh method of approximation. The propagation of plane waves and frequency equations (Othman et al. 2012) were discussed for a thermally conducting linear fibre-reinforced composite materials. Abbas (2011) studied the two-dimensional problem for a fibre-reinforced anisotropic thermoelastic half-space with energy dissipation and compared the results with without energy dissipation. The amplitude and energy ratios of the reflected waves [18] due to incident longitudinal wave are obtained for the plane free boundary in a fibre-reinforced thermoelastic half-space.

The subject of elastic wave propagation is very helpful in determining the characteristics of the material body. It is very common in the field of earthquake engineering, Geophysics, Seismology, etc. There are many open literatures related with the problems of wave propagation in fibre-reinforced composite materials and notable among them are Abbas and Abd-alla (2011); Bedford and Sutherland (1973); Cates and Edwards (1984), Chattopadhyay and Co-workers (2007, 2012); Chen et al. (2011); Green (1991); Othman et al. (2012); Singh (2007); Singh and Tomar (2007b), Zorammuana and Singh (2015a, b) and Tomar and Singh (2006).

Keeping the importance of elastic wave propagation, we have attempted the problem of reflection and refraction of elastic waves due to incident qP and qSV-waves at a plane interface between the homogeneous/inhomogeneous fibre-reinforced elastic half-spaces. Two directional fibre-orientation is used and this makes the problem more
interesting. The reflection and refraction coefficients with energy ratios of the reflected and refracted $qP$ and $qSV$-waves are obtained. These coefficients are computed with the help of MatLab for a particular model.

**BASIC EQUATIONS**

The constitutive relations for a linearly fibre-reinforced elastic medium are given by Belfield et al. (1983) as

$$
\tau_{ij} = \lambda \varepsilon_{ij} + 2\mu \varepsilon_{ij} + \alpha (a_i a_m \varepsilon_{km} + e_{ij} a_i a_j) + 2(\mu_e - \mu) (a_i a_k \varepsilon_{km} + e_{ij} a_i a_j) + \beta a_i a_m e_{km} a_j, \ (i,j,k,m = 1,2,3)
$$

(1)

where $\tau_{ij}$ are stress tensors and $e_{ij}$ are strain tensors with $e_{ij} = \frac{1}{2}(u_{ij} + u_{ji})$, $\lambda$, $\mu_e$ and $\mu$ are elastic constants, $\alpha$ and $\beta$ are specific stress components depending upon the concrete part of the composite materials, $\delta_{ij}$ is Kronecker delta and $\alpha = \{ (a_1, a_2, a_3); a_1^2 + a_2^2 + a_3^2 = 1 \}$.

The equation of motion in a fibre-reinforced elastic material in the absence of body force is given by

$$
\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ij}}{\partial x_j}, \ (i,j = 1,2,3)
$$

(2)

where $u_i = (u_x, u_y, u_z)$ and $(x_1, x_2, x_3) = (x, y, z)$.

We take two dimensional wave propagation in $xz$-plane with the preferred direction of fibre-reinforcement as $(a_1, 0, a_3)$. Eq. (2) may be re-written as

$$
\frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{13}}{\partial z} = \rho \frac{\partial^2 u_1}{\partial t^2},
$$

(3)

$$
\frac{\partial \tau_{21}}{\partial x} + \frac{\partial \tau_{23}}{\partial z} = \rho \frac{\partial^2 u_2}{\partial t^2},
$$

(4)

$$
\frac{\partial \tau_{31}}{\partial x} + \frac{\partial \tau_{33}}{\partial z} = \rho \frac{\partial^2 u_3}{\partial t^2},
$$

(5)

where $\tau_{ij} = (\lambda + 2\mu_e + 2\alpha a_i^2 + 4(\mu_e - \mu) a_i^2 + \beta a_i^3) \frac{\partial u_i}{\partial x} + (\lambda + \alpha + \beta a_i^2 a_j) \frac{\partial u_i}{\partial z} + (\lambda - \mu_e) a_i a_l a_j a_l + \beta a_i a_j \frac{\partial u_i}{\partial x} + \frac{\partial u_l}{\partial z}$,

These are the equations of motion for the homogeneous and inhomogeneous fibre-reinforced elastic materials without body force.

**PROBLEM FORMULATION**

Let $x$ and $y$-axes of the Cartesian coordinate system be on the horizontal plane and the $z$-axis be pointing vertically downward. Consider a homogeneous fibre-reinforced elastic half-space $\{H; z \geq 0 \}$ with $\lambda$, $\alpha$, $\beta$, $\mu_e$, $\mu$ as elastic constants and as density; and an inhomogeneous fibre-reinforced elastic half-space $\{H'; z \leq 0 \}$ with $\lambda'$, $\alpha'$, $\beta'$, $\mu'_e$, $\mu'$ as elastic constants and $\rho'$ as density separated by $z = 0$. The elastic constants and density in the inhomogeneous fibre-reinforced elastic half-space, $H'$ may be defined as [Chatopadhyay and Singh, 2012]

$$
\mu'_e = \mu^{(0)}_e (1 - \varepsilon \cos sz), \quad \mu' = \mu^{(0)} (1 - \varepsilon \cos sz), \quad \rho' = \rho^{(0)} (1 - \varepsilon \cos sz),
$$

$$
\alpha' = \alpha^{(0)} (1 - \varepsilon \cos sz), \quad \beta' = \beta^{(0)} (1 - \varepsilon \cos sz), \quad \lambda' = \lambda^{(0)} (1 - \varepsilon \cos sz),
$$

where $\varepsilon$ is small positive constant and $s$ is real depth parameter.

The equations of motion for $qP$ and $qSV$-waves in the homogeneous fibre-reinforced elastic medium, $H$ are
Zorammuana and Singh

\[ R_1 \frac{\partial^2 u_1}{\partial x^2} + R_2 \frac{\partial^2 u_2}{\partial x \partial z} + R_3 \frac{\partial^2 u_3}{\partial z^2} + R_4 \frac{\partial^2 u_4}{\partial x^2} + R_5 \frac{\partial^2 u_5}{\partial x \partial z} + R_6 \frac{\partial^2 u_6}{\partial z^2} = \rho \frac{\partial^2 u_1}{\partial t^2}, \]  

\[ + R_7 \frac{\partial^2 u_7}{\partial x^2} + R_8 \frac{\partial^2 u_8}{\partial z^2} = \rho \frac{\partial^2 u_1}{\partial t^2}, \]  

(6)

Similarly, the equations of motion for \( qP \) and \( qSV \)-waves in the inhomogeneous fibre-reinforced elastic medium, \( H' \) are given by

\[ R_1^{(0)} (1 - \epsilon \cos sz) \frac{\partial^2 u_1'}{\partial x^2} + R_2^{(0)} (1 - \epsilon \cos sz) \frac{\partial^2 u_2'}{\partial x \partial z} + R_3^{(0)} (1 - \epsilon \cos sz) \frac{\partial^2 u_3'}{\partial z^2} + R_4^{(0)} (1 - \epsilon \cos sz) \frac{\partial^2 u_4'}{\partial x^2} + R_5^{(0)} (1 - \epsilon \cos sz) \frac{\partial^2 u_5'}{\partial x \partial z} + R_6^{(0)} (1 - \epsilon \cos sz) \frac{\partial^2 u_6'}{\partial z^2} = \rho_0 (1 - \epsilon \cos sz) \frac{\partial^2 u_1'}{\partial t^2}, \]  

\[ + R_7^{(0)} (1 - \epsilon \cos sz) \frac{\partial^2 u_7'}{\partial x^2} + R_8^{(0)} (1 - \epsilon \cos sz) \frac{\partial^2 u_8'}{\partial z^2} + \epsilon \sin sz \frac{\partial u'_1}{\partial x} + \epsilon \sin sz \frac{\partial u'_1}{\partial z} = \rho_0 (1 - \epsilon \cos sz) \frac{\partial^2 u_1'}{\partial t^2}, \]  

(9)

where

\[ R_1^{(0)} = \lambda + 2 \mu_1 + 2 \alpha \lambda \mu_1 + 4(\mu_1 - \mu_1)\alpha \lambda \mu_1 + \beta \alpha \lambda \mu_1, \]

\[ R_2^{(0)} = 2[\alpha \lambda \mu_1 a_{13} + 2(\mu_1 - \mu_1)\alpha \lambda \mu_1 a_{13} + \beta \alpha \lambda \mu_1 a_{13}], \]

\[ R_3^{(0)} = \mu_1 + \beta \alpha \lambda \mu_1 a_{13}, \]

\[ R_4^{(0)} = \alpha \lambda \mu_1 a_{13} + 2(\mu_1 - \mu_1)\alpha \lambda \mu_1 a_{13} + \beta \alpha \lambda \mu_1 a_{13}, \]

\[ R_5^{(0)} = \lambda + \alpha + \mu_1 + 2 \beta \alpha \lambda \mu_1 a_{13}, \]

\[ R_6^{(0)} = \alpha \lambda \mu_1 a_{13} + 2(\mu_1 - \mu_1)\alpha \lambda \mu_1 a_{13} + \beta \alpha \lambda \mu_1 a_{13}, \]

\[ R_7^{(0)} = 2[\alpha \lambda \mu_1 a_{13} + 2(\mu_1 - \mu_1)\alpha \lambda \mu_1 a_{13} + \beta \alpha \lambda \mu_1 a_{13}], \]

\[ R_8^{(0)} = \lambda + 2 \mu_1 + 2 \alpha \lambda \mu_1 + 4(\mu_1 - \mu_1)\alpha \lambda \mu_1 + \beta \alpha \lambda \mu_1. \]

\[ \text{WAVE PROPAGATION} \]

Suppose a plane wave (\( qP \) or \( qSV \)-wave) with amplitude, \( A_0 \), making an angle \( \theta_0 \) with the normal be incident at the plane interface, \( z = 0 \) between homogeneous and inhomogeneous fibre-reinforced elastic half-spaces. The incident plane wave gives rise to \( qP \) and \( qSV \)-waves in the half-space, \( H' \) and refracted \( qP \) and \( qSV \)-waves in the half-space, \( H \).

The total displacement components due to incident and reflected waves in the half space, \( H' \) is given by

\[ u_i = \sum_{j=0}^{\infty} A_j \exp[ik_j(z - x \sin \theta_j + x \cos \theta_j)], \]

\[ u_r = \sum_{j=0}^{\infty} \eta_j A_j \exp[ik_j(z - x \sin \theta_j + x \cos \theta_j)], \]

(10)

(11)

where ‘+’ sign corresponds for the incident plane wave and ‘-’ sign corresponds for the reflected waves, \( \omega = k \cdot c \) is the angular frequency, \( A_j \) are the amplitude constants at angles \( \theta_j \) and expressions of the coupling constants \( \eta_0 \), and \( \eta_j \) are given as
Elastic Waves in a Homogeneous/an Inhomogeneous Fibre

\( \eta_0 = -\frac{(R \sin^2 \theta + R \sin \theta \cos \theta + R \cos^2 \theta)}{R \sin^2 \theta + R \sin \theta \cos \theta + R \cos^2 \theta - \rho c^2} \),

\( \eta_j = -\frac{(R \sin^2 \theta_j - R \sin \theta_j \cos \theta_j + R \cos^2 \theta_j)}{R \sin^2 \theta_j - R \sin \theta_j \cos \theta_j + R \cos^2 \theta_j - \rho c^2} \).

Similarly, the displacement components in the half space, \( H' \) are given by

\[ u'_i = \sum_j A_j \exp[i(c_j t - (x \sin \theta_j + z \cos \theta_j))], \quad (12) \]

\[ u'_i = \sum_j \eta_j A_j \exp[i(c_j t - (x \sin \theta_j + z \cos \theta_j))], \quad (13) \]

where \( A_j \) are the amplitude constants at angles \( \theta_j \) and expressions of the coupling constants, \( \eta_j \), may be written as

\[ \eta_j = -\frac{(R \sin^2 \theta_j + R \cos^2 \theta_j)}{R \sin^2 \theta_j + R \cos^2 \theta_j - \rho c^2}. \]

The phase velocities of the reflected and refracted waves are given by

\[ \rho c_{1,2}^2 = \frac{U \pm \sqrt{U^2 - 4V}}{2}, \quad (14) \]

\[ \rho c_{3,4}^2 = \frac{U^{(0)} \pm \sqrt{U^{(0)^2} - 4V^{(0)}}}{2}, \quad (15) \]

where ‘+’ sign corresponds for the \( qP \) wave, while ‘−’ sign corresponds for the \( qSV \) wave and the expressions of \( U, V, U^{(0)} \) and \( V^{(0)} \) are given by

\[ U = R_1 p_1^2 + R_2 p_1 p_3 + R_3 p_1 p_3 + R_4 p_3^2, \]

\[ V = R_1 R_2 p_1^4 + R_1 R_2 p_1^2 p_3 + R_1 R_2 p_1 p_3^2 + R_3 p_1^2 p_3 + \]

\[ + R_3 R_4 p_1^2 p_3 + R_3 R_4 p_1 p_3^2 + R_4 p_3^2 p_3 + \]

\[ + R_4 R_6 p_1^2 p_3 + R_4 R_6 p_1 p_3^2 + (R_1^2 p_1^4 + 2R_4 R_5 p_1^2 p_3 + \]

\[ + 2R_4 R_6 p_1^2 p_3^2 + R_3^2 p_1^4 + 2R_4 R_5 p_3^2 p_3 + \]

\[ + 2R_4 R_6 p_3^2 p_3^2 + 2R_3 R_4 p_1^2 p_3 + R_3^2 p_1^4 + R_3^2 p_3^2), \]

\[ U^{(0)} = R_1^{(0)} p_1^2 + R_2^{(0)} p_1 p_3 + R_3^{(0)} p_1 p_3 + R_4^{(0)} L + R_4^{(0)} M + \]

\[ + R_6^{(0)} M + R_6^{(0)} p_1 p_3 + R_4^{(0)} (p_3^2 + L), \]

\[ V^{(0)} = R_1^{(0)} R_2^{(0)} p_1^3 M + R_1^{(0)} R_2^{(0)} p_1^4 + R_1^{(0)} R_2^{(0)} p_1 p_3 + \]

\[ + R_1^{(0)} R_3^{(0)} (p_1^2 p_3 + p_1^2 L) + R_2^{(0)} R_6^{(0)} p_1 p_3 M + \]

\[ + R_2^{(0)} R_3^{(0)} p_1 p_3 + R_2^{(0)} R_7^{(0)} p_3^2 p_3 + \]

\[ + R_2^{(0)} R_8^{(0)} (p_1 p_3 + p_1 p_3 L) + R_3^{(0)} (p_3^2 p_3 + p_3^2 L) + \]

\[ + R_4^{(0)} R_5^{(0)} (p_1^2 p_3 + p_1^2 L) + R_4^{(0)} R_5^{(0)} p_1 p_3 L + \]

\[ + R_4^{(0)} R_6^{(0)} p_1 p_3 M + R_4^{(0)} R_6^{(0)} (p_3^2 M + LM) - R_4^{(0)} R_5^{(0)} p_1 p_3 M - \]

\[ - R_5^{(0)} R_6^{(0)} M^2 - R_5^{(0)} p_3^4 - 2R_5^{(0)} R_5^{(0)} p_3^2 p_3 + \]

\[ - 2R_4^{(0)} R_6^{(0)} (p_1^2 p_3 + p_1^2 L) - R_4^{(0)} R_4^{(0)} p_1^2 M - R_4^{(0)} R_4^{(0)} p_3^2 p_3 - \]

\[ - 2R_4^{(0)} R_5^{(0)} (p_1 p_3 + p_1 p_3 L) - R_4^{(0)} R_4^{(0)} p_1 p_3 M - \]

\[ - R_6^{(0)} (p_3^2 + L)^2 - R_6^{(0)} R_9^{(0)} (p_3^2 M + LM), \]

\[ L = \frac{i\varepsilon p_3 \sin sz}{(1 - \varepsilon \cos sz)k}, \quad M = \frac{i\varepsilon p_3 \sin sz}{(1 - \varepsilon \cos sz)k}. \]

The Snell’s law of this problem is given by

\[ \sin \theta_q = \sin \theta_q = \sin \theta_q = \sin \theta_q = \sin \theta_q. \]

\[ \frac{c_0}{c_1} = \frac{c_0}{c_2} = \frac{c_0}{c_3} = \frac{c_0}{c_4}, \quad (14) \]

**BOUNDARY CONDITIONS**

The boundary conditions are the continuity of the displacement components and stress tensors at the interface, \( z = 0 \).

(i) continuity of displacements

\[ u_t = u_1', \quad u_3 = u_3', \quad \text{(15)} \]

(ii) continuity of the stress tensors

\[ \tau_{33} = \tau_{33}', \quad \tau_{13} = \tau_{13}'. \]

The continuity of stress tensors may be written in terms of displacement components as

\[ R_8 \frac{\partial u_3}{\partial z} - q_1 \frac{\partial u_3}{\partial x} + R_6 \left( \frac{\partial u_3}{\partial z} + \frac{\partial u_3}{\partial x} \right) = (1 - \varepsilon), \]

\[ \left[ R_9^{(0)} \frac{\partial u'}{\partial z} + R_8^{(0)} \frac{\partial u'}{\partial x} + R_6^{(0)} \left( \frac{\partial u'}{\partial z} + \frac{\partial u'}{\partial x} \right) \right], \quad (17) \]

\[ R_6 \frac{\partial u_3}{\partial z} - m_1 \frac{\partial u_3}{\partial x} + R_3 \left( \frac{\partial u_3}{\partial z} + \frac{\partial u_3}{\partial x} \right) = (1 - \varepsilon), \]

\[ \left[ R_4^{(0)} \frac{\partial u'}{\partial z} + R_3^{(0)} \frac{\partial u'}{\partial x} + R_6^{(0)} \frac{\partial u'}{\partial x} \right]. \quad (18) \]
Using Eqs. (10)-(13) and (16) into the boundary conditions (17) and (18), we have

\[ B Z = D, \]

where

\[
B = \begin{pmatrix}
    b_{11} & b_{12} & b_{13} & b_{14} \\
    b_{21} & b_{22} & b_{23} & b_{24} \\
    b_{31} & b_{32} & b_{33} & b_{34}
\end{pmatrix}, \quad
Z = \begin{pmatrix}
    A_1/A_0 \\
    A_2/A_0 \\
    A_3/A_0
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
    -1 & -1 & -\Delta \\
    -1 & -\Delta
\end{pmatrix},
\]

and

\[
b_{11} = \frac{\eta_1}{\eta_0}, \quad b_{12} = \frac{\eta_1}{\eta_0}, \quad b_{13} = -\frac{\eta_3}{\eta_0}, \quad b_{14} = -\frac{\eta_4}{\eta_0},
\]

\[
b_{21} = q_1k_0 \sin \theta_0 + R_0\eta\eta_3 k_3 \cos \theta_3 - R_0\eta\eta_4 k_4 \cos \theta_4,
\]

\[
b_{22} = q_1k_0 \sin \theta_0 + R_0\eta\eta_2 k_2 \cos \theta_2 + R_0\eta\eta_3 k_3 \cos \theta_3 - R_0\eta\eta_4 k_4 \cos \theta_4,
\]

\[
b_{23} = (1 - \varepsilon)(R^{(0)}_0 k_0 \sin \theta_0 + R^{(0)}_3 \eta \eta_3 k_3 \cos \theta_3 + R^{(0)}_6 \eta \eta_4 k_4 \cos \theta_4),
\]

\[
b_{24} = (1 - \varepsilon)(R^{(0)}_0 k_0 \sin \theta_0 + R^{(0)}_3 \eta \eta_3 k_3 \cos \theta_3 + R^{(0)}_6 \eta \eta_4 k_4 \cos \theta_4),
\]

\[
b_{31} = m_1k_0 \sin \theta_0 + R_1 k_1 \cos \theta_1 - R_3\eta\eta_3 k_3 \sin \theta_3 + R_4\eta\eta_4 k_4 \sin \theta_4,
\]

\[
b_{32} = m_1k_0 \sin \theta_0 + R_1 k_1 \cos \theta_1 - R_3\eta\eta_3 k_3 \sin \theta_3 + R_4\eta\eta_4 k_4 \sin \theta_4,
\]

\[
b_{33} = (1 - \varepsilon)(R^{(0)}_1 k_0 \sin \theta_0 + R^{(0)}_3 \eta \eta_3 k_3 \cos \theta_3 + R^{(0)}_6 \eta \eta_4 k_4 \cos \theta_4),
\]

\[
b_{34} = (1 - \varepsilon)(R^{(0)}_1 k_0 \sin \theta_0 + R^{(0)}_3 \eta \eta_3 k_3 \cos \theta_3 + R^{(0)}_6 \eta \eta_4 k_4 \cos \theta_4),
\]

\[
\Delta = q_1k_0 \sin \theta_0 - R_0\eta\eta_3 k_3 \cos \theta_0 - R_0\eta\eta_4 k_4 \cos \theta_0 - R_0\eta\eta_4 k_4 \sin \theta_0.
\]

These equations will help to find the reflection and refraction coefficients of the reflected and refracted \( qP \) and \( qSV \)-waves.

**SOLUTION OF THE PROBLEM**

With the help of Eq. (19), we obtain the amplitude constants of the reflected and refracted waves as

\[
\frac{A_1}{A_0} = \frac{D_1}{D_0} = \frac{A_2}{A_0} = \frac{D_2}{D_0} = \frac{A_3}{A_0} = \frac{D_3}{D_0} = \frac{A_4}{A_0} = \frac{D_4}{D_0},
\]

where

\[
D_0 = \begin{pmatrix}
    1 & 1 & -1 & -1 \\
    b_{11} & b_{12} & b_{13} & b_{14} \\
    b_{21} & b_{22} & b_{23} & b_{24} \\
    b_{31} & b_{32} & b_{33} & b_{34}
\end{pmatrix},
\]

\[
D_1 = \begin{pmatrix}
    -1 & 1 & -1 & -1 \\
    b_{11} & b_{12} & b_{13} & b_{14} \\
    -\Delta & b_{22} & b_{23} & b_{24} \\
    -\Delta & b_{32} & b_{33} & b_{34}
\end{pmatrix},
\]

\[
D_2 = \begin{pmatrix}
    b_{11} & -1 & b_{13} & b_{14} \\
    b_{21} & -\Delta & b_{23} & b_{24} \\
    b_{31} & -\Delta & b_{33} & b_{34}
\end{pmatrix},
\]

\[
D_3 = \begin{pmatrix}
    b_{11} & b_{12} & -1 & b_{14} \\
    b_{21} & b_{22} & -\Delta & b_{24} \\
    b_{31} & b_{32} & -\Delta & b_{34}
\end{pmatrix},
\]

\[
D_4 = \begin{pmatrix}
    1 & 1 & -1 & -1 \\
    b_{11} & b_{12} & b_{13} & b_{14} \\
    b_{21} & b_{22} & b_{23} & -\Delta \\
    b_{31} & b_{32} & b_{33} & -\Delta
\end{pmatrix}.
\]
Elastic Waves in a Homogeneous/an Inhomogeneous Fibre

Using the amplitude constants corresponding to incident displacement components, \( u_i \) and \( u_j \), the amplitude of the incident wave is given by

\[
\sqrt{A_0^2 + \eta_0^2 A_0^2} = \sqrt{1 + \eta_0^2 A_0^2}.
\]

Similarly, the amplitude of the reflected and refracted waves are given by \( \sqrt{1 + \eta_j^2 A_j} \) \((j = 1, 2, 3, 4)\). Thus, reflection coefficients \((r_{ij}, r_{ij}')\) and refraction coefficients \((t_{ij}, t_{ij}')\) are given by

\[
r_{ij} = \frac{1 + \eta_j^2 A_j}{1 + \eta_0^2 A_0}, \quad r_{ij}' = \frac{1 + \eta_j^2 A_j}{1 + \eta_0^2 A_0}, \quad (21)
\]

\[
t_{ij} = \frac{1 + \eta_j^2 A_j}{1 + \eta_0^2 A_0}, \quad t_{ij}' = \frac{1 + \eta_j^2 A_j}{1 + \eta_0^2 A_0}. \quad (22)
\]

The energy transmission per unit area at the interface, \( z = 0 \) may be written as

\[
E^* = \tau_{3j} \cdot \dot{u}_i + \tau_{3i} \cdot \dot{u}_j + \tau_{3j}' \cdot \dot{u}_i' + \tau_{3i}' \cdot \dot{u}_j'. \quad (23)
\]

The expression of the energy due to incident wave is given by

\[
E_{inc} = -k_0 \omega J_0 A_0^2 \exp[2i(\omega t - k_0(x \sin \theta_0 + z \cos \theta_0))], \quad (24)
\]

where

\[
J_0 = \{q_1 \sin \theta_0 - R_0 \eta_0 \cos \theta_0 - R_0 (\cos \theta_0 + \eta_0 \sin \theta_0)\} \eta_0 + m \sin \theta_0 - R_0 (\cos \theta_0 - \eta_0 \sin \theta_0),
\]

\[
J_0 = \{q_1 \sin \theta_0 + R_0 \eta_0 \cos \theta_0 + R_0 (\cos \theta_0 - \eta_0 \sin \theta_0)\} \eta_0 + m \sin \theta_0 + R_0 (\cos \theta_0 + \eta_0 \sin \theta_0).
\]

The modulus of energy ratios of the reflected and refracted \( qP \) and \( qSV \)-waves are given as

\[
E_1 = \frac{k_1 J_1}{k_0 J_0} \frac{A_1}{A_0}, \quad E_2 = \frac{k_2 J_2}{k_0 J_0} \frac{A_2}{A_0}, \quad (25)
\]

\[
E_3 = \frac{k_3 J_3}{k_0 J_0} \frac{A_3}{A_0}, \quad E_4 = \frac{k_4 J_4}{k_0 J_0} \frac{A_4}{A_0},
\]

where the expressions of \( J_1, J_2, J_3 \) and \( J_4 \) are given by

\[
J_1 = \{q_1 \sin \theta_1 + R_1 \eta_1 \cos \theta_1 + R_1 (\cos \theta_1 - \eta_1 \sin \theta_1)\} \eta_1 + m_1 \sin \theta_1 + R_1 (\cos \theta_1 + \eta_1 \sin \theta_1) + R_1 \eta_1 \cos \theta_1,
\]

\[
J_2 = \{q_1 \sin \theta_2 + R_1 \eta_2 \cos \theta_2 + R_1 (\cos \theta_2 - \eta_2 \sin \theta_2)\} \eta_2 + m_1 \sin \theta_2 + R_1 (\cos \theta_2 + \eta_2 \sin \theta_2) + R_1 \eta_2 \cos \theta_2,
\]

\[
J_3 = \{q_1 \sin \theta_3 + R_1 \eta_3 \cos \theta_3 + R_1 (\cos \theta_3 - \eta_3 \sin \theta_3)\} \eta_3 + m_1 \sin \theta_3 + R_1 (\cos \theta_3 + \eta_3 \sin \theta_3) + R_1 \eta_3 \cos \theta_3,
\]

\[
J_4 = \{1 - \varepsilon\} \{R_0^{(0)} \eta_4 \cos \theta_4 + \eta_4 \sin \theta_4\} \eta_4 + m_1 \sin \theta_4 + R_1^{(0)} (\cos \theta_4 + \eta_4 \sin \theta_4) + R_1^{(0)} \eta_4 \cos \theta_4.
\]

We have seen that the energy ratios are functions of the amplitude ratios, elastic constants, fibre orientation, inhomogeneity parameter and the angle of incidence. We have observed that these coefficients depend on the elastic constants, fibre orientation, inhomogeneity parameter and the angle of incidence.

If \( qP \)-wave is incident, then \( \theta = \theta', c = c' \) and the reflection coefficients \((r_{iP}, r_{iP}')\) and refraction coefficients \((t_{iP}, t_{iP}')\) of the reflected and refracted \( qP \) and \( qSV \)-waves are given by Eqs. (21) and (22) with \( i = p \). Moreover, if the incident wave is \( qSV \)-wave, then \( \theta = \theta' \), \( c_0 = c_2 \) and the reflection coefficients \((r_{iP}, r_{iP}')\) and refraction coefficients \((t_{iP}, t_{iP}')\) of the reflected and refracted \( qP \) and \( qSV \)-waves are given by Eqs. (21) and (22) with \( i = s \). Eq. (25) gives their energy ratios of the reflected and refracted \( qP \) and \( qSV \)-waves.

**PARTICULAR CASES**

**Case I:** If the half-space, \( H' \) reduces to homogeneous fibre-reinforced elastic medium, the problem reduces to the reflection and refraction of elastic waves at the plane interface between the two dissimilar homogeneous fibre-reinforced elastic half-spaces. With \( \varepsilon = 0 \) and the phase velocities of \( qP \) and \( qSV \)-waves in the half space, \( H' \) are given by Eq. (15) with the following modified values

\[
U^{(0)} = R_1^{(0)} p_1^2 + R_2^{(0)} p_1 p_3 + R_3^{(0)} p_3 + R_4^{(0)} p_3^2,
\]

\[
V^{(0)} = R_1^{(0)} R_3^{(0)} p_1^2 + R_4^{(0)} R_3^{(0)} p_3^2 + R_4^{(0)} R_5^{(0)} p_1^2 p_3 + R_2^{(0)} R_3^{(0)} p_1^3 + R_2^{(0)} R_5^{(0)} p_3^2 + R_2^{(0)} R_4^{(0)} p_3^3.
\]
The reflection and refraction coefficients and energy ratios corresponding to the reflected and refracted $qP$ and $qSV$-waves are given by Eqs. (21)-(22) and (25) with the following changes

$$c_1^2 = \frac{2\mu + \lambda}{\rho}, \quad c_2^2 = \frac{2\mu_0 + \lambda^{(0)}}{\rho_0},$$

$$c_3^2 = \frac{\mu}{\rho}, \quad c_4^2 = \frac{\mu_0}{\rho_0},$$

which are the results of classical elasticity (Achenbach, 1976).

The reflection and refraction coefficients and energy ratios of the reflected and refracted $qP$ and $qSV$-waves are given by Eqs. (21)-(22) and (25) respectively with

$$R_{21} = -\lambda k_0 \sin \theta_0 + (\lambda + 2\mu)\eta k_0 \cos \theta_0,$$

$$R_{22} = -\lambda k_0 \sin \theta_0 + (\lambda + 2\mu)\eta k_0 \cos \theta_0,$$

$$R_{23} = (\lambda^{(0)} k_0 \sin \theta_0 + (\lambda^{(0)} + 2\mu)\eta k_0 \cos \theta_0),$$

$$R_{24} = (\lambda^{(0)} k_0 \sin \theta_0 + (\lambda^{(0)} + 2\mu)\eta k_0 \cos \theta_0),$$

$$b_{31} = \mu(k_1 \cos \theta_1 - \eta k_0 \sin \theta_0),$$

$$b_{32} = \mu(k_2 \cos \theta_2 - \eta k_0 \sin \theta_0),$$

$$b_{33} = \mu_0(k_3 \cos \theta_3 + \eta k_0 \sin \theta_0),$$

$$b_{34} = \mu_0(k_4 \cos \theta_4 + \eta k_0 \sin \theta_0),$$

$$\Delta = -\lambda k_0 \sin \theta_0 - (\lambda + 2\mu)\eta k_0 \cos \theta_0,$$

$$\Delta' = -\mu(k_0 \cos \theta_0 + \eta k_0 \sin \theta_0),$$

$$\eta_0 = \frac{-(\lambda + \mu) \sin \theta_0 \cos \theta_0}{\mu \sin^2 \theta_0 + (\lambda + 2\mu) \cos^2 \theta_0 - \rho_0^2},$$

$$\eta_j = \frac{(\lambda + \mu) \sin \theta_0 \cos \theta_0}{\mu \sin^2 \theta_0 + (\lambda + 2\mu) \cos^2 \theta_0 - \rho_j^2}, \quad (j = 1, 2),$$

$$\eta_j = \frac{-(\lambda^{(0)} + \mu_0) \sin \theta_0 \cos \theta_0}{\mu_0 \sin^2 \theta_0 + (\lambda^{(0)} + 2\mu_0) \cos^2 \theta_0 - \rho_j^2}, \quad (j = 3, 4),$$

$$R_2 = R_3 = R_4 = R_5 = R_6 = R_7 = R_8 = R_9 = 0, \quad q_1 = -\lambda, \quad R_i = \mu, \quad R_i = \lambda + \mu.$$
NUMERICAL RESULTS AND DISCUSSION

For the numerical computation of the reflection and refraction coefficients and energy ratios of the reflected and refracted waves for the incident $qP$ and $qSV$-waves, we take the following relevant parameters.

For the homogeneous half-space, $H$:
- $\lambda = 6.92\times10^9 \text{ N/m}^2$, $\mu_r = 3.05\times10^9 \text{ N/m}^2$,
- $\mu_l = 4.07\times10^9 \text{ N/m}^2$, $\alpha = -0.49\times10^9 \text{ N/m}^2$,
- $\beta = 0.23\times10^9 \text{ N/m}^2$, $\rho_0 = 1260 \text{ Kg/m}^3$.

For the non-homogeneous half space, $H'$:
- $\lambda(0) = 7.59\times10^9 \text{ N/m}^2$, $\mu_r(0) = 1.89\times10^9 \text{ N/m}^2$,
- $\mu_l(0) = 2.45\times10^9 \text{ N/m}^2$, $\alpha(0) = -1.28\times10^9 \text{ N/m}^2$,
- $\beta(0) = 0.32\times10^9 \text{ N/m}^2$, $\rho_0 = 7800 \text{ Kg/m}^3$.

The variation of the modulus of reflection and refraction coefficients with angle of incidence, $\theta_0$ are depicted in Figures 1–4, while those of energy ratios with the angle of incidence are shown in Figures 4–8 for different values of inhomogeneity parameter, $\varepsilon$. In all these figures, (a) will correspond for the incident $qP$-wave and (b) will correspond for the incident $qSV$-wave.

In Figure 1(a), the modulus of the reflection coefficient, $r_{pp}$ corresponding to the reflected $qP$-wave for incident $qP$-wave starts from certain values and increases with the increase of $\theta_0$, attaining the maximum value at the grazing angle of incidence. We observed that the values of $r_{pp}$ decrease with the increase of $\varepsilon$. Figure 1(b) shows the variation of the modulus of reflection coefficient, $r_{sp}$ corresponding to the reflected $qP$-wave for the incident $qSV$-wave. Curves I and II shows that $r_{sp}$ start from certain value at the normal incidence, which increase initially and then decrease upto $\theta_0 = 39^\circ$ which again increase and then decrease to zero value with the changes of $\theta_0$.

In Figure 2(a), the reflection coefficient, $r_{sp}$ corresponding to the reflected $qSV$-wave for incident $qP$-wave increases to the maximum value at the grazing angle of incidence. These coefficients increase with increase of $\varepsilon$. In Figure 2(b), the reflection coefficient, $r_{sp}$ for the incident $qSV$-wave increases with the increase of $\theta_0$ and it attains the maximum value at the grazing angle of incidence. We observed that the minimum value of $r_{sp}$ is at the grazing angle of incidence.

In Figure 3, the refraction coefficients, $t_{pp}$ and $t_{sp}$ increase to the maximum value with the increase of $\theta_0$ which decrease to the minimum value at the grazing angle of incidence. These coefficients increase with the increase of $\varepsilon$. In Figure 4, we observed that the refraction coefficients, $t_{pp}$ and $t_{sp}$ start from certain value and decrease to minimum value with the increase of $\theta_0$.

(b) Reflection Coefficient, $r_{pp}$

Fig. 1: Variation of the Modulus of Reflection Coefficients with Angle of Incidence, $\theta_0$
Fig. 2: Variation of the Modulus of Reflection Coefficients with Angle of Incidence, \( \theta_i \)

(a) Reflection Coefficient, \( R_{pz} \)

(b) Reflection Coefficient, \( R_{zz} \)

Fig. 3: Variation of the Modulus of Refraction Coefficients with Angle of Incidence, \( \theta_i \)

(a) Refraction Coefficient, \( P_{pz} \)

(b) Refraction Coefficient, \( P_{zz} \)

Fig. 4: Variation of the Modulus of Refraction Coefficients with Angle of Incidence, \( \theta_i \)

(a) Refraction Coefficient, \( P_{pp} \)

(b) Refraction Coefficient, \( P_{ss} \)
Elastic Waves in a Homogeneous/an Inhomogeneous Fibre

Fig. 5: Variation of the Modulus of Energy Ratio, $E_1$ with Angle of Incidence, $\theta_0$

(a) Incident qP waves

(b) Incident qSV waves

Fig. 6: Variation of the Modulus of Energy Ratio, $E_2$ with Angle of Incidence, $\theta_0$

(a) Incident qP waves

(b) Incident qSV waves

Fig. 7: Variation of the Modulus of Energy Ratio, $E_3$ with Angle of Incidence, $\theta_0$
In Figure 5(a), the energy ratio, $E_1$, corresponding to the reflected $qP$-wave for incident $qP$-wave starts from certain values and increases upto $\theta_0 = 87^\circ$ and decreases thereafter with the increase of $\theta_0$. The values of $E_1$ for incident $qP$-wave decrease with the increase of $\epsilon$. Figure 5(b) shows the variation of the modulus of $E_1$ for incident $qSV$-wave with the angle of incidence. Curves I and II show that $E_1$ start from certain value which decreases upto $\theta_0 = 31^\circ$ and then increase to the maximum values at $\theta_0 = \epsilon$ for Curve I and $\theta_0$ for Curve II, which decrease thereafter with the increase of $\theta_0$. Curve III shows that $E_1$ increases initially up to certain value of $\theta_0$ which decreases for some values and increases thereafter, up to the maximum value, which again decreases with the increase of $\theta_0$.

In Figure 6(a), the energy ratio, $E_2$, corresponding to the reflected $qSV$-wave for incident $qP$-wave increases initially with the increase of $\theta_0$ and attains a maximum value, thereafter it decreases to zero value at the grazing angle of incidence. The values of $E_2$ for incident $qP$-wave decrease with the increase of $\epsilon$. But, the value of $E_2$ for incident $qSV$-wave increases with the increase of $\theta_0$ in Figure 6(b) and their values increase with the increase of $\epsilon$. In Figure 7, the energy ratio $E_3$ for the incident $qP$ and $qSV$-waves make a parabolic region in $0^\circ \leq \theta_0 \leq 90^\circ$ and their values increase with the increase of $\epsilon$. We have observed that the sum of energy ratios is close to unity.

CONCLUSION

The problem of reflection and refraction of elastic waves due to incident $qP$ and $qSV$-waves at a plane interface between the homogeneous and inhomogeneous fibre-reinforced elastic half-spaces has been investigated. The reflection and refraction coefficients and energy ratios corresponding to the reflected and refracted $qP$ and $qSV$-waves for the incident $qP$ and $qSV$-waves are obtained analytically and numerically.

We may conclude with the following points:

1. The reflection and refraction coefficients and energy ratios corresponding to the reflected and refracted waves for incident $qP$ and $qSV$-waves are functions of elastic constants, fibre orientation, inhomogeneity parameter and angle of incidence.

2. The energy ratios, $E_1$ for incident $qP$-wave, $E_2$ for incident $qSV$-wave and reflection coefficients, $r_{pp}$, attain the maximum value at grazing angle of incidence.

3. The energy ratio $E_2$ for incident $qSV$-wave and the coefficients, $r_{pp}$, increase with the increase of angle of incidence.

4. The energy ratio, $E_4$ and the coefficients, $t_{pp}$, $t_{ps}$, $t_{sp}$, $t_{ss}$ decrease with the increase of $\theta_0$.

5. The values of energy ratios, $E_3$, $E_4$ and the coefficients, $r_{pp}$, $t_{pp}$, $t_{ps}$, $t_{sp}$, $t_{ss}$ increase with the increase of inhomogeneity parameter, $\epsilon$.

6. The sum of energy ratios is close to unity.
ACKNOWLEDGEMENTS

The author (C. Zorammuana) acknowledges UGC, New Delhi for giving financial assistance through Rajiv Gandhi National Fellowship (RGNF) to complete this work.

REFERENCES