CONFERENCE PROCEEDINGS OF SCIENCE AND TECHNOLOGY

Conference Proceedings of Science and Technology

ISSN: 2651-544X

Preface

Dear Conference Participant,

The 18th International Geometry Symposium has been planned to be held face to face in honor of Prof. Dr. Sadık KELEŞ on 01-04 July 2020; but due to the COVID-19 pandemic which affected the whole world, it has been postponed to be held online on July 12-13, 2021. We would like to thank Prof. Dr. Ahmet KIZILAY, the Rector of İnönü University and members of honorary commitee, Prof. Dr. H. Hilmi HACISALİHOĞLU and Prof. Dr. Sadık KELEŞ, who supported us for the 18th Geometry Symposium to be held at our university.

We would also like to thankHead of Mathematics Department at İnönü University Prof. Dr. A. İhsan SİVRİDAĞ, the management and members of Association of Geometers for their interest and support. Continuity in holding an international symposium is a team effort that requires dedication. For this reason, we would like to thank the scientific and organizing committee members of the symposium for all their support. In addition, we would like to thank the invited speakers who participated in our symposium from Turkey and abroad, and all the participants who supported the symposium by participating in the symposium with and without papers. We hope that the studies presented in this symposium will contribute to science and scientists.

Prof. Dr. Rıfat GÜNEŞ
Chairman
On behalf of the Organizing Committee

Contents

Conference Proceeding Science and Technology, 4(2), 2021, 201–205

18th International Geometry Symposium, 2021.

The quaternionic ruled surfaces in terms of Bishop frame

ISSN: 2651-544X
http://cpostjournal.org/

Abdussamet Çalışkan

[1] Accounting and Tax Applications, Fatsa Vocational School, Ordu University, Ordu, Turkey, ORCID: 0000-0002-1512-2452
** Corresponding Author E-mail: abdussamet65@gmail.com*

Abstract: In this paper, we investigate the quaternionic expression of the ruled surfaces drawn by the motion of the Bishop vectors. The distribution parameters, the pitches, and the angle of pitches of the ruled surfaces are calculated as quaternionic.

Keywords: Distribution parameter, Angle of pitch, Bishop Frame, Ruled surface, Spatial quaternion, Quaternion, The pitch.

1 Introduction

The quaternion was discovered in 1843 by Hamilton [1]. Quaternions arose historically from Hamilton's essays in the mid-nineteenth century to generalize complex numbers in some way that would apply to three-dimensional (3D) space. A feature of quaternions is closely related to 3D rotations, a fact apparent to Hamilton almost immediately but first published by Hamilton's contemporary Arthur Cayley in 1845 [2]. The technology did not penetrate the computer animation community until the landmark Siggraph 1985 paper of Ken Shoemake [3]. The importance of Shoemake's paper is that it took the concept of the orientation frame for moving 3D objects and cameras, which require precise orientation specification, exposed the deficiencies of the then-standard Euler-angle methods, and introduced quaternions to animators as a solution. The Serret-Frenet formulae for quaternionic curves in $\mathbb{IR}^3$ and $\mathbb{IR}^4$ were introduced by K. Bharathi and M. Nagaraj [4]. There are lots of studies that investigated quaternionic curves by using this study. One of them is Karadağ and Sivridağ's study whose they gave many characterizations for quaternionic inclined curves in $\mathbb{IR}^4$ [5]. Şenyurt *et al.* calculated curvature and torsion of spatial quaternionic involute curve according to the normal vector and the unit Darboux vector of Smarandache curve [6]. In [7], the authors investigated the ruled surface as spatial quaternionic. They quaternionally calculated the integral invariants of the ruled surface. Bishop frame, which is called alternate or parallel frame of curves depending on parallel vector fields, was defined by Bishop. Thanks to this frame, Bishop frame is used as an alternative roof for situations where the Frenet frame cannot be defined (especially where the second derivative of the curve is zero) [8]. Hanson and Hui investigated Bishop frame as quaternion. They found interesting results [9]. Masal and Azak attempted to introduce ruled surfaces generated from the Bishop vectors. Some properties of integral invariants of these surfaces were discussed and they obtained some important results [10]. Tunçer examined ruled surfaces generated from the Bishop vectors with different method and he obtained some characterizations [11].

A surface is said to be "ruled" if it is generated by moving a straight line continuously in Euclidean space $\mathbb{IR}^3$. Ruled surfaces are one of the simplest objects in geometric modeling. One important fact about ruled surfaces is that they can be generated by straight lines. A practical application of ruled surfaces is that they are used in civil engineering. The result is that if engineers are planning to construct something with curvature, they can use a ruled surface since all the lines are straight. Among ruled surfaces, developable surfaces form an important subclass since they are useful in sheet metal design and processing [12, 13].

In this study, we investigate the ruled surfaces drawn by the motion of the Bishop vectors as quaternionic. We calculate integral invariants of the ruled surface with the theory of quaternion.

2 Preliminaries

In E^3, the standard inner product is given by $\langle x, x\rangle = x_1^2 + x_2^2 + x_3^2$ where $x = (x_1, x_2, x_3) \in E^3$. Let $\alpha : I \to E^3$ be a unit speed curve. Denote by $\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\}$ the moving Frenet frame. $\vec{T}(s)$ is the tangent vector field, $\vec{N}(s)$ is the principal normal vector field and $\vec{B}(s)$ is the binormal vector field of the curve α, respectively. The Frenet formulas are given by [14]

$$\vec{T}'(s) = \kappa(s)\vec{N}(s),\ \vec{N}'(s) = -\kappa(s)\vec{T}(s) + \tau(s)\vec{B}(s),\ \vec{B}'(s) = -\tau(s)\vec{N}(s).$$

Here curvature and torsion of the curve α are defined with [14]

$$\kappa(s) = \|\alpha''(s)\|,\ \tau(s) = \frac{\langle \alpha'(s) \wedge \alpha''(s), \alpha'''(s)\rangle}{\|\alpha'(s) \wedge \alpha''(s)\|^2}.$$

Bishop frame, which is called alternate or parallel frame of curves depending on parallel vector fields, was defined by Bishop. Bishop's equations are similar to the Frenet equations:

$$\begin{bmatrix} T'(s) \\ N_1'(s) \\ N_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{bmatrix}, \tag{1}$$

where $k_1(s)$ and $k_2(s)$ are Bishop curvatures. The formulas between Bishop curvatures and Frenet curvatures are [9, 10]

$$\kappa = \sqrt{k_1^2 + k_2^2}, \;\; \theta = \arctan(\frac{k_2}{k_1}), \;\; \tau = \theta'(s),$$
$$k_1(s) = \kappa(s)\cos\phi(s), \;\; k_2(s) = \kappa(s)\sin\phi(s).$$

On the other hand, Steiner rotation and Steiner translation vectors are

$$D = \oint(-k_2N_1 + k_1N_2)ds, \;\; V = \oint d\alpha,$$

respectively, [10].
Real quaternion is defined by the $1, e_1, e_2, e_3$. 1 is a real number, e_1, e_2, e_3 are vectors with the following properties:

$$\begin{aligned} &{e_1}^2 = {e_2}^2 = {e_3}^2 = e_1 \times e_2 \times e_3 = -1, e_1, e_2, e_3 \in \mathbb{R}^3, \\ &e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = e_2. \end{aligned} \tag{2}$$

The 4-dimensional real Euclidean space $\mathbb{R}^4$ is identified with the space of real quaternions

$$\mathbb{K} = \left\{ q = d + ae_1 + be_2 + ce_3 | a, b, c, d \in \mathbb{R}, \vec{e}_1, e_2, e_3 \in \mathbb{R}^3 \right\}$$

in [4, 15].
Let $q_1 = S_{q_1} + V_{q_1} = d_1 + a_1e_1 + b_1e_2 + c_1e_3$ and $q_2 = S_{q_2} + V_{q_2} = d_2 + a_2e_1 + b_2e_2 + c_2e_3$ be two quaternions in $\mathbb{K}$, the quaternion multiplication of q_1 and q_2 is given by

$$\begin{aligned} q_1 \times q_2 &= d_1d_2 - (a_1a_2 + b_1b_2 + c_1c_2) + (d_1a_2 + a_1d_2 + b_1c_2 - c_1b_2)e_1 \\ &+ (d_1b_2 + b_1d_2 + b_1a_2 - a_1b_2)e_2 + +(d_1c_2 + c_1d_2 + a_1b_2 - b_1a_2)e_3. \end{aligned}$$

The symmetric real-valued bilinear form h which is defined as

$$h : \mathbb{K} \times \mathbb{K} \to \mathbb{R}, \;\; h(q_1, q_2) = \frac{1}{2}(q_1 \times \bar{q_2} + q_2 \times \bar{q_1})$$

is called quaternion inner product [4]. Let q be a real quaternion. Its conjugate is $\bar{q} = S_q - V_q$. The norm of a real quaternion is a real number in the form of

$$N(q) = \sqrt{h(q,q)} = \sqrt{d^2 + a^2 + b^2 + c^2}.$$

If $N(q) = 1$, q is called a unit quaternion. Invers of real quaternion is $q^{-1} = \frac{\bar{q}}{N(q)}\cdot$ Quaternion the division is noncommutative, and is defined by the (order-dependent) relations $r_1 = q_1 \times {q_2}^{-1}, r_2 = {q_2}^{-1} \times q_1$. Where r_1 is the right division, r_2 is the left division [15]. The three-dimensional real Euclidean space $\mathbb{R}^3$ is identified with the space of spatial quaternions

$$Q = \{q \in \mathbb{K} \mid q + \bar{q} = 0\}$$

in the obvious manner [4]. In this case, the elements of Q are $q = ae_1 + be_2 + ce_3$. As a result, the quaternion multiplication of the two spatial quaternions is [15]

$$q_1 \times q_2 = -\langle q_1, q_2 \rangle + q_1 \wedge q_2. \tag{3}$$

Definition 1. Let $s \in .I = [0, 1]$ be the arc parameter along the smooth curve

$$\alpha : [0, 1] \to Q, \;\; \alpha(s) = \sum_{i=1}^{3} \alpha_i(s)e_i.$$

This is called a spatial quaternionic curve [4].
Definition 2. A ruled surface in $\mathbb{R}^3$ is a surface that contains at least one 1-parameter family of straight lines. Thus a ruled surface has a parametrization in the form

$$\varphi : I \times \mathbb{R} \to \mathbb{R}^3, \;\; \overrightarrow{\varphi}(s, v) = \overrightarrow{\alpha}(s) + v\overrightarrow{x}(s),$$

where we call α the anchor curve, X the generator vector of the ruled surface [14].

The quaternionic express of distribution parameter (drall) belonging to the ruled surface is given by [7]

$$P_x = \frac{h(\vec{x}\times\vec{x}',\alpha')}{\mathbf{N}(\vec{x}')^2} = \frac{1}{2}\frac{\Big((\vec{x}\times\vec{x}')\times\overline{\alpha'}+\alpha'\times\overline{(\vec{x}\times\vec{x}')}\Big)}{\mathbf{N}(\vec{x}')^2}. \quad (4)$$

The angle of pitch and the pitch of the closed quaternionic ruled surface, λ_x and L_x, are equal to the projection of the generator x on the Steiner rotation vector $\vec{D}$ and the Steiner translation vector $\vec{V}$ [7]

$$\lambda_x = h(\overrightarrow{D},\overrightarrow{x}), \quad (5)$$

$$L_x = h(\overrightarrow{V},\overrightarrow{x}). \quad (6)$$

3 The Quaternionic Ruled Surfaces in terms of Bishop Frame

The ruled surfaces drawn by the motion of the Bishop vectors are given by

$$\begin{aligned}\varphi_T(s,v) &= \vec{\alpha}(s)+v\vec{T}(s),\\ \varphi_{N_1}(s,v) &= \vec{\alpha}(s)+v\vec{N_1}(s),\\ \varphi_{N_2}(s,v) &= \vec{\alpha}(s)+v\vec{N_2}(s).\end{aligned}$$

Using the equation 4, the distribution parameter of the closed spatial quaternionic ruled surface drawn by the motion of the principal vectors $\vec{T}$ belonging to the spatial quaternionic curve α is

$$P_T = \frac{h(\vec{T}\times\vec{T}',\vec{\alpha}')}{\mathbf{N}(\vec{T}')^2}. \quad (7)$$

Considering the equations 1 and 3, we obtain

$$\begin{aligned}h(\vec{T}\times\vec{T}',\vec{T}) &= \frac{1}{2}\Big((\vec{T}\times\vec{T}')\times\overline{\vec{T}}+\vec{T}\times(\overline{\vec{T}\times\vec{T}'})\Big)\\ &= \frac{1}{2}\Big((\vec{T}\times(k_1\vec{N_1}+k_2\vec{N_2}))\times\overline{\vec{T}}+\vec{T}\times(\overline{\vec{T}\times(k_1\vec{N_1}+k_2\vec{N_2})})\Big)\\ &= \frac{1}{2}\Big((k_1(\vec{T}\times\vec{N_1})+k_2(\vec{T}\times\vec{N_2}))\times\overline{\vec{T}}+\vec{T}\times(\overline{k_1(\vec{T}\times\vec{N_1})+k_2(\vec{T}\times\vec{N_2})})\Big)\\ &= \frac{1}{2}\Big((k_1(-\langle\vec{T},\vec{N_1}\rangle+\vec{T}\wedge\vec{N_1})+k_2(-\langle\vec{T},\vec{N_2}\rangle+\vec{T}\wedge\vec{N_2}))\times\overline{\vec{T}}\\ &\quad+\vec{T}\times(\overline{k_1(-\langle\vec{T},\vec{N_1}\rangle+\vec{T}\wedge\vec{N_1})+k_2(-\langle\vec{T},\vec{N_1}\rangle+\vec{T}\wedge\vec{N_1})})\Big)\\ &= \frac{1}{2}\Big((k_1\vec{N_2}-k_2\vec{N_1})\times\overline{\vec{T}}+\vec{T}\times(\overline{k_1\vec{N_2}-k_2\vec{N_1}})\Big)\\ &= 0.\end{aligned}$$

If this value is substituted in equation 7, $P_T = 0$ is found.
Similarly, the distribution parameter of the closed spatial quaternionic ruled surface drawn by the motion of the vector $\vec{N_1}$ is

$$P_{N_1} = \frac{h(\vec{N_1}\times\vec{N_1}',\vec{\alpha}')}{\mathbf{N}(\vec{N_1}')^2}. \quad (8)$$

Considering the equations 1 and 3, we can write

$$\begin{aligned}h(\vec{N_1}\times\vec{N_1}',\vec{T}) &= \frac{1}{2}\Big((\vec{N_1}\times\vec{N_1}')\times\overline{\vec{T}}+\vec{T}\times(\overline{\vec{N_1}\times\vec{N_1}'})\Big)\\ &= \frac{1}{2}\Big((\vec{N_1}\times(-k_1\vec{T}))\times\overline{\vec{T}}+\vec{T}\times(\overline{\vec{N_1}\times(-k_1\vec{T})})\Big)\\ &= \frac{1}{2}\Big(-(\vec{N_1}\times\vec{T})\times k_1(\vec{T}\times\vec{T})-(\vec{T}\times\vec{N_1})\times(k_1(\vec{T}\times\vec{T}))\Big)\\ &= 0.\end{aligned}$$

If this value is substituted in equation 8, $P_{N_1} = 0$ is found.

Similarly, the distribution parameter of the closed spatial quaternionic ruled surface drawn by the motion of the vector $\vec{N_2}$ is

$$P_{N_2} = \frac{h(\vec{N_2} \times \vec{N_2}', \vec{\alpha}')}{\mathbf{N}(\vec{N_2}')^2}. \tag{9}$$

Considering the equations 1 and 3, we can write

$$\begin{aligned}
h(\vec{N_2} \times \vec{N_2}', \vec{T}) &= \frac{1}{2}\Big((\vec{N_2} \times \vec{N_2}') \times \overline{\vec{T}} + \vec{T} \times \overline{(\vec{N_2} \times \vec{N_2}')}\Big) \\
&= \frac{1}{2}\Big((\vec{N_2} \times (-k_2\vec{T})) \times \overline{\vec{T}} + \vec{T} \times \overline{(\vec{N_2} \times (-k_2\vec{T}))}\Big) \\
&= \frac{1}{2}\Big(-(\vec{N_2} \times \vec{T}) \times k_2(\vec{T} \times \vec{T}) - (\vec{T} \times \vec{N_2}) \times (k_2(\vec{T} \times \vec{T}))\Big) \\
&= 0.
\end{aligned}$$

If this value is substituted in equation 9, $P_{N_2} = 0$ is found.

Corollary 1. *The ruled surfaces drawn by the motion of the Bishop vectors are developable as quaternionic.*

Theorem 1. *The pitches of the closed spatial quaternionic ruled surfaces drawn by the Bishop vectors are*

$$L_T = \oint ds, \; L_{N_1} = L_{N_2} = 0.$$

Proof: According to the equations 3 and 6, the pitches of the closed spatial quaternionic ruled surfaces drawn by the motion of the Bishop vectors belonging to the spatial quaternionic curve α are as follows:

$$\begin{aligned}
L_T &= h(\oint d\vec{\alpha}, \vec{T}) = h(\oint \vec{T} ds, \vec{T}) = \frac{1}{2}\Big(\vec{T}\oint ds \times \overline{\vec{T}} + \vec{T} \times \overline{\vec{T}\oint ds}\Big) \\
&= \oint ds, \\
L_{N_1} &= h(\oint d\vec{\alpha}, \vec{N_1}) = h(\oint \vec{T} ds, \vec{N_1}) = \frac{1}{2}\Big(\vec{T}\oint ds \times \overline{\vec{N_1}} + \vec{N_1} \times \overline{\vec{T}\oint ds}\Big) \\
&= \frac{1}{2}\Big(\big(\vec{T} \times (-N_1)\big)\oint ds + \big(N_1 \times (-\vec{T})\big)\oint ds\Big) \\
&= 0, \\
L_{N_2} &= h(\oint d\vec{\alpha}, \vec{N_2}) = h(\oint \vec{T} ds, \vec{N_2}) = \frac{1}{2}\Big(\vec{T}\oint ds \times \overline{\vec{N_2}} + \vec{N_2} \times \overline{\vec{T}\oint ds}\Big) \\
&= 0.
\end{aligned}$$

□

Theorem 2. *The angle of pitches of the closed spatial quaternionic ruled surfaces drawn by the Bishop vectors are*

$$\lambda_T = 0, \;\; \lambda_{N_1} = -\oint k_2 ds, \;\; \lambda_{N_2} = \oint k_1 ds.$$

Proof: According to the equations 3 and 5, the angles of pitches of the closed spatial quaternionic ruled surfaces drawn by the motion of the Bishop vectors belonging to the spatial quaternionic curve α are as follows:

$$\begin{aligned}
\lambda_T &= h(\vec{D}, \vec{T}) = \frac{1}{2}\Big(\vec{D} \times \overline{\vec{T}} + \vec{T} \times \overline{\vec{D}}\Big) \\
&= \frac{1}{2}\left(\left(\oint(-k_2N_1 + k_1N_2)ds\right) \times \overline{\vec{T}} + \vec{T} \times \overline{\left(\oint(-k_2N_1 + k_1N_2)ds\right)}\right) \\
&= \frac{1}{2}\Big(\vec{N_1} \times \vec{T}\oint k_2 ds - \vec{N_2} \times \vec{T}\oint k_1 ds + \vec{T} \times \vec{N_1}\oint k_2 ds - \vec{T} \times \vec{N_2}\oint k_1 ds)\Big) \\
&= \frac{1}{2}\Big(-\vec{N_2}\oint k_2 ds - \vec{N_1}\oint k_1 ds + \vec{N_2}\oint k_2 ds + \vec{N_1}\oint k_1 ds)\Big) \\
&= 0,
\end{aligned}$$

$$
\begin{aligned}
\lambda_{N_1} &= h(\vec{D},\vec{N_1}) = \frac{1}{2}\Big(\vec{D}\times\overline{\vec{N_1}} + \vec{N_1}\times\overline{\vec{D}}\Big)\\
&= \frac{1}{2}\left(\left(\oint(-k_2N_1+k_1N_2)ds\right)\times\overline{\vec{N_1}} + \vec{N_1}\times\overline{\left(\oint(-k_2N_1+k_1N_2)ds\right)}\right)\\
&= \frac{1}{2}\Big((\vec{N_1}\times\vec{N_1})\oint k_2ds - (\vec{N_2}\times\vec{N_1})\oint k_1ds + (\vec{N_1}\times\vec{N_1})\oint k_2ds\\
&\quad -(\vec{N_1}\times\vec{N_2})\oint k_1ds)\Big)\\
&= -\oint k_2ds,
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{N_2} &= h(\vec{D},\vec{N_2}) = \frac{1}{2}\Big(\vec{D}\times\overline{\vec{N_2}} + \vec{N_2}\times\overline{\vec{D}}\Big)\\
&= \frac{1}{2}\left(\left(\oint(-k_2N_1+k_1N_2)ds\right)\times\overline{\vec{N_2}} + \vec{N_2}\times\overline{\left(\oint(-k_2N_1+k_1N_2)ds\right)}\right)\\
&= \frac{1}{2}\Big((\vec{N_1}\times\vec{N_2})\oint k_2ds - (\vec{N_2}\times\vec{N_2})\oint k_1ds + (\vec{N_2}\times\vec{N_1})\oint k_2ds\\
&\quad -(\vec{N_2}\times\vec{N_2})\oint k_1ds)\Big)\\
&= \oint k_1ds.
\end{aligned}
$$

□

4 Conclusion

The distribution parameter, pitch, and angle of pitch are the invariants in the ruled surface. We quaternionically express the ruled surface drawn by the Bishop vectors. These invariants are quaternionically calculated for ruled surfaces.

Acknowledgement

We would like to thank the reviewers for their detailed comments and suggestions for the manuscript.

5 References

1 W. R. Hamilton, *Elements of quaternions*, Longmans, Green Company, 1866.
2 A. Cayley, *On Certain Results Relating to Quaternions*, Phil. Mag., **26** (1845), 141–145.
3 K. Shoemake, *Animating Rotation with Quaternion Curves*, Siggraph Computer Graphics, **19** (1985), 245–254.
4 K. Bharathi, M. Nagaraj, *Quaternion Valued Function of a Real Variable Serret-Frenet Formulae*, Ind. J. P. Appl. Math., **18** (1987), 507–511.
5 M. Karadağ, A. İ. Sivridağ, *Quaternion Valued Functions of a Single reel Variable and Inclined Curves*, Journal of Natural Science, **13**(1-2) (1997), 23-36.
6 S. Şenyurt, C. Cevahir, Y. Altun, *On spatial quaternionic involute curve a new view*, Adv. Appl. Clifford Algebr., **27**(2) (2016), 1815–1824.
7 S. Şenyurt, A. Çalışkan, *The Quaternionic Expression of Ruled Surfaces*, Filomat, **32**(16) (2018), 5753–5766 .
8 L. R Bishop, *There is more than one way to frame a curve*, The American Mathematical Monthly, **82**(3) (1975), 246–251.
9 A. J. Hanson, M. Hui, *Quaternion frame approach to streamline visualization*, IEEE Transactions on Visualization and Computer Graphics, **1**(2) (1995), 164-174.
10 M. Masal, A.Z. Azak, *Ruled surfaces according to Bishop frame in the Euclidean 3-space*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, **89**(2) (2019), 415-424.
11 Y. Tunçer, *Ruled surfaces with the Bishop frame in Euclidean 3-space*, Gen Math Notes, **26** (2015), 74-83.
12 K. Orbay, E. Kasap, İ. Aydemir, *Mannheim Offsets of Ruled Surfaces*, Math. Probl. Eng., **2009** (2009), 9 pages.
13 A. T. Ali, *Non-lightlike ruled surfaces with constant curvatures in Minkowski 3-space*, Int. J. Geom. Methods Mod. Phys., **15**(4) (2017), 1850068.
14 M. P. Do Carmo, *Differential Geometry of Curves and Surfaces* (Revised Second Edition), Prentice-Hall, Mineola, New York, 2016.
15 H. H. Hacısalihoğlu, *Motion Geometry and Quaternions Theory* (in Turkish), Faculty of Sciences and Arts, University of Gazi Press, 1983.

Conference Proceeding Science and Technology, 4(2), 2021, 206–214

18th International Geometry Symposium, 2021.

A New Look on Oresme Numbers: Dual-Generalized Complex Component Extension

ISSN: 2651-544X
http://cpostjournal.org/

Gülsüm Yeliz ŞENTÜRK[1,⋆] Nurten GÜRSES[2] Salim YÜCE[3]

[1] Department of Computer Engineering, Faculty of Engineering and Architecture, Istanbul Gelisim University, Istanbul, Turkey, ORCID:0000-0002-8647-1801
[2] Department of Mathematics, Faculty of Arts and Sciences, Yildiz Technical University, Istanbul, Turkey, ORCID:0000-0001-8407-854X
[3] Department of Mathematics, Faculty of Arts and Sciences, Yildiz Technical University, Istanbul, Turkey, ORCID:0000-0002-8296-6495
** Corresponding Author E-mail: gysenturk@gelisim.edu.tr*

Abstract: Our main interest in this paper is to explore dual-generalized complex ($\mathcal{DGC}$) Oresme sequence extension. We present two new types of Oresme numbers. We investigate special linear recurrence relations and summation properties for $\mathcal{DGC}$ Oresme numbers of type-1. Furthermore, we describe the recurrence relation of $\mathcal{DGC}$ Oresme numbers of type-1 in matrix form. We also discuss the theory using the doubling approach to $\mathcal{DGC}$ Oresme sequence and then investigate all of the notions for type-2.

Keywords: Dual-generalized complex numbers, Oresme numbers.

1 Literature Review

Recurrence sequences attract much interest and have been a central part of number theory for a long time now. Moreover, these sequences appear almost everywhere, not only in mathematics but also in physics, engineering, cryptography, biology, economics, computer algorithms, and in our daily lives, for instance, the interest portion of monthly payments made to pay off something.

Horadam sequence $W_n\,(a,b;p,q)$, where a,b,p,q are arbitrary integers and $W_0=a, W_1=b$, so named after the papers of A. F. Horadam, is a special linear recurrence sequence*. With special specific values of $(a,b;p,q)$, the Horadam sequence reduces to the Fibonacci, generalized Fibonacci, Lucas, generalized Lucas, Pell, Pell-Lucas, modified Pell, Jacobsthal, Jacobsthal-Lucas, Mersenne, Fermat, balancing, Lucas-balancing, and so on. Many direct or indirect publications on this sequence have appeared in the literature, i.e., [1–10].

Furthermore, one can extend the values of p,q in Horadam sequence to be arbitrary rational numbers. In the mid-fourteenth century, French philosopher and naturalist N. Oresme studied on the sum of the sequence of rational numbers:

$$\frac{1}{2},\frac{2}{4},\frac{3}{8},\frac{4}{16},\frac{5}{32},\frac{6}{64},\frac{7}{128},\frac{8}{256},\ldots.$$

Unfortunately, Oresme's original working papers remained unpublished. The derivation of Oresme sequence from the general Horadam sequence (for $a=0, b=\frac{1}{2}, p=1, q=\frac{1}{4}$), were discussed in [12], as a special case. The properties of Oresme numbers were investigated by Horadam's paper [12]. The biological role of this special sequence is an answer to the question, "if we know the first two terms, i.e., the proportion of grandparents and parents of different genotypes, how do we calculate the proportions in any later generations?" as L. Hogben remark [13].

In 2004, C. K. Cook presented the generalization of Oresme numbers in more than one way and established identities analogous to Horadam's, [14]. In 2019, T. Goy considered some families of Toeplitz-Hessenberg determinants, the entries of which are Oresme numbers. These determinant formulas were rewritten as identities involving the sums of products of Oresme numbers and multinomial coefficients, [15]. In 2019, G. Cerda-Morales studied the generalization of Oresme numbers with a new sequence of numbers called Oresme polynomials. Moreover, using the matrix methods for Oresme polynomials, the identities, including the general bilinear index-reduction formula, were obtained. Finally, Oresme polynomials that are natural extensions of the $k-$Oresme numbers were introduced and investigated, [16]. In 2020,

**Some of the special cases of Horadam sequence can be found in OEIS, [11]. More specifically,*

- *A085939 for Horadam sequence* $W_n(0,1;6,4)$,
- *A085449 for Horadam sequence* $W_n(0,1;4,2)$,
- *A085504 for Horadam sequence* $W_n(0,1;9,3)$,
- *A000045 for Fibonacci sequence* $W_n(0,1;1,-1)$,
- *A000032 for Lucas sequence* $W_n(2,1;1,-1)$,
- *A000129 for Pell sequence* $W_n(0,1;2,-1)$,
- *A273692 for Oresme sequence* $W_n\left(0,\frac{1}{2};1,\frac{1}{4}\right)$.

the characteristic equation of each Fibonacci, Lucas, Mersenne, Oresme, Jacobsthal, Pell, Leonardo, Padovan, Perrin and Narayana sequence was given, and then their respective roots were investigated and analyzed, through fractal theory based on Newton's method, [17]. For that, Google Colab was used as a technological tool, [17]. In 2021, the authors extended the generalization of the matrix form of Oresme sequence to the field of integers. In addition, the hybrid Oresme sequence was introduced, and mathematical properties and theorems were obtained, [18]. In other respect, 2-component number systems can be classified as follows:

- complex numbers $\mathbb{C}$ with elements $z = a + bi$, $i^2 = -1$, [19],
- hyperbolic (double, binary, split complex, perplex) numbers $\mathbb{H}$ with elements $z = a + bj$, $j^2 = 1, j \neq \pm 1$, [20–22],
- dual numbers $\mathbb{D}$ with elements, $z = a + b\varepsilon$, $\varepsilon^2 = 0, \varepsilon \neq 0$, [20, 23, 24].

The generalization of the above systems is the set of generalized-complex numbers:

$$\mathbb{C}_{\mathfrak{p}} := \left\{ z = a + bJ : \ a, b \in \mathbb{R}, \ J^2 = \mathfrak{p}, \, \mathfrak{p} \in \mathbb{R}, J \notin \mathbb{R} \right\}$$

examined in [25, 26], which led to the construction of other n-dimensional number systems. $\mathbb{C}_{\mathfrak{p}}$ is a vector space over $\mathbb{R}$. It is an analog to complex numbers $\mathbb{C}$ for $\mathfrak{p} = -1$, hyperbolic numbers $\mathbb{H}$ for $\mathfrak{p} = 1$ and dual numbers $\mathbb{D}$ for $\mathfrak{p} = 0$.

Additionally, some of the four-component number systems which can be constructed by utilizing the complex, hyperbolic and dual two-component systems are:

- complex-hyperbolic numbers (or hyperbolic-complex) in [25, 27–30],
- complex-dual numbers (or dual-complex) in [27, 31–33],
- dual-hyperbolic numbers in [27, 28],
- bicomplex numbers, as an extension of complex numbers, in [34–37],
- bihyperbolic numbers, as an extension of hyperbolic numbers, in [37–41],
- hyper-dual numbers, as an extension of dual numbers, in [42–44].

Taking into account all of these and using the Cayley-Dickson doubling procedure for construction, the dual-generalized complex ($\mathcal{DGC}$) are investigated in [45], considering various properties and matrix representations. The set of $\mathcal{DGC}$ numbers are defined as [45]:

$$\mathbb{DC}_{\mathfrak{p}} := \left\{ \tilde{a} = z_1 + z_2\varepsilon : \ z_1, z_2 \in \mathbb{C}_{\mathfrak{p}}, \ \varepsilon^2 = 0, \, \varepsilon \neq 0, \varepsilon \notin \mathbb{R} \right\}.$$

It is analog to dual-complex numbers for $\mathfrak{p} = -1$, dual-hyperbolic numbers for $\mathfrak{p} = 1$ and hyperdual numbers for $\mathfrak{p} = 0$.

With the help of dual numbers and dual-complex numbers, dual-complex generalized $k-$Horadam numbers have been carefully scrutinized in [46]. Hyper-dual Horadam numbers have been studied, and the characteristic identities for Horadam numbers have been presented in [47]. The Horadam Hybrid numbers and the relations about them have been examined in [48, 49]. The Binet formula and the generating function of bicomplex Horadam numbers have been described, and two important identities that relate the matrix theory to the second-order recurrence relations are obtained [50]. The dual-hyperbolic Horadam numbers and their well-known identities have been discussed in [51]. Finally, the $\mathcal{DGC}$ Horadam numbers are defined and the relations about them are presented in [52].

In this present study, we are interested in the following three problems.

Problem 1. Is it possible to extend the Oresme sequence for the $\mathcal{DGC}$ numbers?

Problem 2. If the answer to Problem 1 is affirmative, what relations and properties are satisfied?

Problem 3. Is it possible to define a new $\mathcal{DGC}$ Oresme sequence by using a doubling process?

This paper is organized as follows: Section 2 presents general information for Oresme sequence, $\mathcal{DGC}$ numbers and $\mathcal{DGC}$ Horadam sequence. Section 3 and Section 4 are the main contributions of this paper. In Section 3, we attempt to extend some related computational results about Oresme sequence to $\mathcal{DGC}$ Oresme sequence. Furthermore, within the framework of the doubling process, we define another $\mathcal{DGC}$ Oresme sequence and discuss its relations in Section 4. In the conclusion section, we reduce $\mathcal{DGC}$ Oresme sequence to dual complex, hyper-dual and dual-hyperbolic Oresme sequences.

2 Basic Notations and Arguments

First, let us recall $\mathcal{DGC}$ numbers. Then also recall Oresme sequence as a special case of the Horadam sequence introducing some properties. Additionally, considering the paper [52], we present major elements of the $\mathcal{DGC}$ Horadam sequence.

2.1 $\mathcal{DGC}$ Numbers

The $\mathcal{DGC}$ numbers are of the form [45]:

$$\tilde{a} = z_1 + z_2\varepsilon = a_1 + a_2J + a_3\varepsilon + a_4J\varepsilon.$$

The base elements $\{1, J, \varepsilon, J\varepsilon\}$ satisfy the conditions as follows:

$$J^2 = \mathfrak{p}, \quad (J\varepsilon)^2 = 0, \quad J\varepsilon = \varepsilon J. \tag{1}$$

The operations for $\mathcal{DGC}$ numbers are given as follows, respectively:

- equality: $\tilde{a}_1 = \tilde{a}_2 \Leftrightarrow z_{11} = z_{21}, z_{12} = z_{22}$,

- addition (and hence subtraction): $\tilde{a}_1+\tilde{a}_2=(z_{11}+z_{21})+(z_{12}+z_{22})\varepsilon$,
- scalar multiplication: $\lambda\tilde{a}_1=\lambda z_{11}+\lambda z_{12}\varepsilon$,
- multiplication: $\tilde{a}_1\tilde{a}_2=(z_{11}z_{21})+(z_{11}z_{22}+z_{12}z_{21})\varepsilon$,

where $\tilde{a}_1=z_{11}+z_{12}\varepsilon, \tilde{a}_2=z_{21}+z_{22}\varepsilon\in\mathbb{DC}_{\mathfrak{p}}$ and $\lambda\in\mathbb{R}$. $\mathbb{DC}_{\mathfrak{p}}$ is a commutative ring with unity and a vector space over real numbers. For further information, we refer the reader to [45].

2.2 Horadam Sequence and Its Special Types

Definition 1. *For $a,b,p,q\in\mathbb{Z}$, the generalized sequence of $W_n(a,b;p,q)$, briefly W_n, is satisfied in the following second-order recurrence relation*

$$W_n(a,b;p,q)=pW_{n-1}-qW_{n-2}, \tag{2}$$

where initial conditions are $W_0=a$, $W_1=b$. In honor of Horadam, this general sequence is called a Horadam sequence, [1–3].

Definition 2. *The n-th Oresme sequence, briefly O_n, is satisfied in the following second-order recurrence relation, [12]:*

$$O_n=O_{n-1}-\frac{1}{4}O_{n-2},\ \forall n\in\mathbb{Z}, \tag{3}$$

where initial conditions are $O_0=0$, $O_1=\frac{1}{2}$, that is special kind of Horadam sequence obtained by taking $a=0, b=\frac{1}{2}, p=1, q=\frac{1}{4}$ in equation (2)*.*

Remark 1. $\forall n\in\mathbb{N}$*, the following recurrence relation can also be written for Oresme numbers with negative subscripts (see in [18]):*

$$O_{-n}=4O_{-n+1}-4O_{-n+2}. \tag{4}$$

Hence we can write several values of Oresme numbers as follows:

...	O_{-6}	O_{-5}	O_{-4}	O_{-3}	O_{-2}	O_{-1}	O_0	O_1	O_2	O_3	O_4	O_5	O_6	...
...	-384	-160	-64	-24	-8	-2	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{5}{32}$	$\frac{3}{32}$	...

Also, the characteristic equation of Oresme sequence is determined by $x^2-x+\frac{1}{4}=0$.

Theorem 1. *The permutation of rows and columns of the $\mathbf{O}$ generating matrix can be performed by obtaining another matrix from Oresme sequence, totaling the following matrices for positive and negative integer terms as follows (see details in [16, 18]):*

- *For* $\mathbf{O}=\begin{bmatrix}1 & -\frac{1}{4}\\ 1 & 0\end{bmatrix}$ *and $n\geq 1$, we have* $\mathbf{O}^n=\begin{bmatrix}2O_{n+1} & -\frac{1}{2}O_n\\ 2O_n & -\frac{1}{2}O_{n-1}\end{bmatrix}$ *in [18].*
- *For* $\mathbf{O}=\begin{bmatrix}0 & 1\\ -\frac{1}{4} & 1\end{bmatrix}$ *and $n\geq 1$, we have* $\mathbf{O}^n=\begin{bmatrix}-\frac{1}{2}O_{n-1} & 2O_n\\ -\frac{1}{2}O_n & 2O_{n+1}\end{bmatrix}$ *in [18].*
- *For* $\mathbf{O}=\begin{bmatrix}0 & 1\\ -4 & 4\end{bmatrix}$ *and $n>0$, we have* $\mathbf{O}^n=\begin{bmatrix}2O_{-n+1} & -\frac{1}{2}O_{-n}\\ 2O_{-n} & -\frac{1}{2}O_{-n-1}\end{bmatrix}$ *in [18].*
- *For* $\mathbf{O}=\begin{bmatrix}4 & -4\\ 1 & 0\end{bmatrix}$ *and $n>0$, we have* $\mathbf{O}^n=\begin{bmatrix}-\frac{1}{2}O_{-n-1} & 2O_{-n}\\ -\frac{1}{2}O_{-n} & 2O_{-n+1}\end{bmatrix}$ *in [18].*

Proposition 1. *Let O_n be n-th Oresme number. Then the following properties hold:*

1. $O_n=\frac{1}{2}U_{n-1}$*, where* $U_n=W_n\left(1,1;1,\frac{1}{4}\right)$,
2. $\lim_{n\to\infty}O_n=0$,
3. $\lim_{n\to-\infty}O_n=-\infty$,
4. $\lim_{n\to\infty}\dfrac{O_n}{O_{n-1}}=\dfrac{1}{2}$,
5. $O_n=\dfrac{n}{2^n},\ n\in\mathbb{Z}$
6. $O_nO_{-n}=-n^2$,
7. $\dfrac{O_{-n}}{O_n}=-2^{2n}$,
8. $O_{n+2}-\dfrac{3}{4}O_n+\dfrac{1}{4}O_{n-1}=0$,
9. $O_{n+2}-\dfrac{3}{4}O_{n+1}+\dfrac{1}{16}O_{n-1}=0$,
10. $\sum_{j=0}^{n-1}O_j=4\left(\dfrac{1}{2}-O_{n+1}\right)$,
11. $\sum_{j=0}^{\infty}O_j=2$,
12. $\sum_{j=0}^{n-1}(-1)^jO_j=\dfrac{4}{9}\left(-\dfrac{1}{2}+(-1)^n\left(O_{n+1}-2O_n\right)\right)$,
13. $\sum_{j=0}^{n-1}O_{2j}=\dfrac{4}{9}\left(2+O_{2n-1}-5O_{2n}\right)$,
14. $\sum_{j=0}^{n-1}O_{2j+1}=\dfrac{1}{9}(10+5O_{2n-1}-16O_{2n})$.

Definition 3. *The $\mathcal{DGC}$ Horadam sequence $\tilde{\mathcal{W}}_n(a,b;p,q)$ is defined by:*

$$\tilde{\mathcal{W}}_n=W_n+W_{n+1}J+W_{n+2}\varepsilon+W_{n+3}J\varepsilon$$

and satisfy the following recurrence relation:

$$\tilde{\mathcal{W}}_n=p\tilde{\mathcal{W}}_{n-1}-q\tilde{\mathcal{W}}_{n-2},\ \ (n\geq 2).$$

For details, we refer the reader to [52].

$\mathcal{DGC}$ Horadam numbers can be exactly examined in Table 1 as follow:

$\tilde{\mathcal{W}}_n\,(a,b;p,q)$	$\mathcal{DGC}$ **Horadam sequence**
$\tilde{\mathcal{W}}_n\,(0,1;1,-1)$	$\mathcal{DGC}$ Fibonacci sequence
$\tilde{\mathcal{W}}_n\,(0,1;p,q)$	$\mathcal{DGC}$ Generalized Fibonacci sequence
$\tilde{\mathcal{W}}_n\,(2,1;1,-1)$	$\mathcal{DGC}$ Lucas sequence
$\tilde{\mathcal{W}}_n\,(2,p;p,q)$	$\mathcal{DGC}$ Generalized Lucas sequence
$\tilde{\mathcal{W}}_n\,(0,1;2,-1)$	$\mathcal{DGC}$ Pell sequence
$\tilde{\mathcal{W}}_n\,(2,2;2,-1)$	$\mathcal{DGC}$ Pell-Lucas sequence
$\tilde{\mathcal{W}}_n\,(1,1;2,-1)$	$\mathcal{DGC}$ Modified Pell sequence
$\tilde{\mathcal{W}}_n\,(0,1;1,-2)$	$\mathcal{DGC}$ Jacobsthal sequence
$\tilde{\mathcal{W}}_n\,(2,1;1,-2)$	$\mathcal{DGC}$ Jacobsthal-Lucas sequence
$\tilde{\mathcal{W}}_n\,(0,1;3,2)$	$\mathcal{DGC}$ Mersenne sequence
$\tilde{\mathcal{W}}_n\,(1,3;3,-2)$	$\mathcal{DGC}$ Fermat sequence
$\tilde{\mathcal{W}}_n\,(0,1;6,1)$	$\mathcal{DGC}$ balancing sequence
$\tilde{\mathcal{W}}_n\,(1,3;6,1)$	$\mathcal{DGC}$ Lucas-balancing sequence

Table 1 Major $\mathcal{DGC}$ Horadam sequences

3 $\mathcal{DGC}$ Oresme Numbers of Type-1

In this original section, we extend the familiar relations of Oresme numbers to $\mathcal{DGC}$ Oresme numbers.

Definition 4. *Considering $a=0, b=\frac{1}{2}, p=1, q=\frac{1}{4}$ in $\mathcal{DGC}$ Horadam sequence $\tilde{\mathcal{W}}_n\,(a,b;p,q)$, and denoting a term of this special sequence by $\tilde{\mathcal{O}}_n$, we get $\mathcal{DGC}$ Oresme sequence. The n-th $\mathcal{DGC}$ Oresme number of type-1 is of the form:*

$$\tilde{\mathcal{O}}_n = O_n + O_{n+1}J + O_{n+2}\varepsilon + O_{n+3}J\varepsilon, \tag{5}$$

where O_n is the n-th Oresme number. The $\mathcal{DGC}$ Oresme numbers of type-1 satisfy the following second-order relation

$$\tilde{\mathcal{O}}_{n+2} = \tilde{\mathcal{O}}_{n+1} - \frac{1}{4}\tilde{\mathcal{O}}_n,\ \forall n \in \mathbb{Z}.$$

The followings are several values of $\tilde{\mathcal{O}}_n$:

$$\begin{aligned}
\tilde{\mathcal{O}}_{-3} &= -24 - 8J - 2\varepsilon & \tilde{\mathcal{O}}_1 &= \frac{1}{2} + \frac{1}{2}J + \frac{3}{8}\varepsilon + \frac{1}{4}J\varepsilon,\\
\tilde{\mathcal{O}}_{-2} &= -8 - 2J + \frac{1}{2}J\varepsilon & \tilde{\mathcal{O}}_2 &= \frac{1}{2} + \frac{3}{8}J + \frac{1}{4}\varepsilon + \frac{5}{32}J\varepsilon,\\
\tilde{\mathcal{O}}_{-1} &= -2 + \frac{1}{2}\varepsilon + \frac{1}{2}J\varepsilon & \tilde{\mathcal{O}}_3 &= \frac{3}{8} + \frac{1}{4}J + \frac{5}{32}\varepsilon + \frac{3}{32}J\varepsilon,\\
\tilde{\mathcal{O}}_0 &= \frac{1}{2}J + \frac{1}{2}\varepsilon + \frac{3}{8}J\varepsilon, & \tilde{\mathcal{O}}_4 &= \frac{1}{4} + \frac{5}{32}J + \frac{3}{32}\varepsilon + \frac{7}{128}J\varepsilon.
\end{aligned}$$

We give a fundamental approach to define elementary arithmetic operations. Consider the $\mathcal{DGC}$ Oresme numbers of type-1 $\tilde{\mathcal{O}}_n$ and $\tilde{\mathcal{O}}_m$. Equality is as follows:

$$\tilde{\mathcal{O}}_n = \tilde{\mathcal{O}}_m \Leftrightarrow O_n = O_m\ \wedge\ O_{n+1} = O_{m+1}\ \wedge\ O_{n+2} = O_{m+2}\ \wedge\ O_{n+3} = O_{m+3}.$$

Addition (and hence subtraction) of $\tilde{\mathcal{O}}_n$ to another Oresme $\tilde{\mathcal{O}}_m$ acts in a component-wise way:

$$\tilde{\mathcal{O}}_n + \tilde{\mathcal{O}}_m = (O_n + O_m) + (O_{n+1} + O_{m+1})J + (O_{n+2} + O_{m+2})\varepsilon + (O_{n+3} + O_{m+3})J\varepsilon.$$

The scalar multiplication of $\tilde{\mathcal{O}}_n$ with a scalar λ gives another $\mathcal{DGC}$ Oresme number

$$\lambda\tilde{\mathcal{O}}_n = \lambda O_n + \lambda O_{n+1}J + \lambda O_{n+2}\varepsilon + \lambda O_{n+3}J\varepsilon,\ \ \lambda \in \mathbb{R}.$$

Multiplication of the $\mathcal{DGC}$ Oresme number is performed as

$$\begin{aligned}\tilde{\mathcal{O}}_n\tilde{\mathcal{O}}_m = \; & O_nO_m + \mathfrak{p}O_{n+1}O_{m+1} + (O_{n+1}O_m + O_nO_{m+1})\,J + (O_nO_{m+2} + O_{n+2}O_m + \mathfrak{p}(O_{n+1}O_{m+3} + O_{n+3}O_{m+1}))\,\varepsilon \\ & + (O_{n+1}O_{m+2} + O_nO_{m+3} + O_{n+3}O_m + O_{n+2}O_{m+1})\,J\varepsilon.\end{aligned}$$

One can write the recurrence relation of $\mathcal{DGC}$ Oresme numbers of type-1 for $n \geq 1$ in matrix form as the following:

$$\begin{bmatrix}\hat{\mathcal{O}}_{n+1}\\ \hat{\mathcal{O}}_n\end{bmatrix} = \begin{bmatrix}1 & -\frac{1}{4}\\ 1 & 0\end{bmatrix}\begin{bmatrix}\hat{\mathcal{O}}_n\\ \hat{\mathcal{O}}_{n-1}\end{bmatrix}.$$

If we consider $\hat{\mathsf{O}}_n = \begin{bmatrix}\hat{\mathcal{O}}_{n+1}\\ \hat{\mathcal{O}}_n\end{bmatrix}$ and $\mathbf{O} = \begin{bmatrix}1 & -\frac{1}{4}\\ 1 & 0\end{bmatrix}$, then we get linear system $\hat{\mathsf{O}}_n = \mathbf{O}\hat{\mathsf{O}}_{n-1}$. More generally, we have the following theorem.

Theorem 2. *The solution of the system $\hat{\mathsf{O}}_n = \mathbf{O}\hat{\mathsf{O}}_{n-1}$ can also be given in terms of the powers of the $\mathbf{O}$-matrix. That is*

$$\hat{\mathsf{O}}_n = \mathbf{O}^n\hat{\mathsf{O}}_0,$$

where $\hat{\mathsf{O}}_0 = \begin{bmatrix}\frac{1}{2} + \frac{1}{2}J + \frac{3}{8}\varepsilon + \frac{1}{4}J\varepsilon\\ \frac{1}{2}J + \frac{1}{2}\varepsilon + \frac{3}{8}J\varepsilon\end{bmatrix}$ *is the initial solution and* $\mathbf{O}^n = \begin{bmatrix}2O_{n+1} & -\frac{1}{2}O_n\\ 2O_n & -\frac{1}{2}O_{n-1}\end{bmatrix}$ *(see details $\mathbf{O}$ and $\mathbf{O}^n$ in [16, 18]).*

Remark 2. *According to different matrix forms of Oresme sequence in Theorem 1, one can give another version of Theorem 2.*

The different conjugations and modules can be defined in Table 2 :

Conjugates	Modules
$\tilde{\mathcal{O}}_n^{\dagger_1} = (O_n - O_{n+1}J) + (O_{n+2} - O_{n+3}J)\,\varepsilon$	$\mathcal{N}_{\tilde{\mathcal{O}}_n}^{\dagger_1} = \tilde{\mathcal{O}}_n\tilde{\mathcal{O}}_n^{\dagger_1}$
$\tilde{\mathcal{O}}_n^{\dagger_2} = (O_n + O_{n+1}J) - (O_{n+2} + O_{n+3}J)\,\varepsilon$	$\mathcal{N}_{\tilde{\mathcal{O}}_n}^{\dagger_2} = \tilde{\mathcal{O}}_n\tilde{\mathcal{O}}_n^{\dagger_2}$
$\tilde{\mathcal{O}}_n^{\dagger_3} = (O_n - O_{n+1}J) - (O_{n+2} - O_{n+3}J)\,\varepsilon$	$\mathcal{N}_{\tilde{\mathcal{O}}_n}^{\dagger_3} = \tilde{\mathcal{O}}_n\tilde{\mathcal{O}}_n^{\dagger_3}$

Table 2 Conjugations and modules of $\mathcal{DGC}$ Oresme numbers of type-1

Bearing in mind Definition 4, Table 2 and Proposition 1, the following mathematical expressions can be given:

Proposition 2. *Let $\tilde{\mathcal{O}}_n$ be n-th $\mathcal{DGC}$ Oresme number. Then, the below properties can be given:*

- $\tilde{\mathcal{O}}_n + \tilde{\mathcal{O}}_n^{\dagger_1} = 2\,(O_n + O_{n+2}\varepsilon) = \frac{1}{2^n}\left(2n + \frac{n+2}{2}\varepsilon\right)$,
- $\tilde{\mathcal{O}}_n\tilde{\mathcal{O}}_n^{\dagger_1} = O_n^2 - \mathfrak{p}O_{n+1}^2 + 2\,(O_nO_{n+2} - \mathfrak{p}O_{n+1}O_{n+3})\,\varepsilon$,
- $\tilde{\mathcal{O}}_n + \tilde{\mathcal{O}}_n^{\dagger_2} = 2\,(O_n + O_{n+1}J) = \frac{1}{2^n}(2n + (n+1)J)$,
- $\tilde{\mathcal{O}}_n\tilde{\mathcal{O}}_n^{\dagger_2} = O_n^2 + \mathfrak{p}O_{n+1}^2 + 2O_nO_{n+1}J$,
- $\tilde{\mathcal{O}}_n + \tilde{\mathcal{O}}_n^{\dagger_3} = 2\,(O_n + O_{n+3}J\varepsilon) = \frac{1}{2^n}\left(2n + \frac{n+3}{4}J\varepsilon\right)$,
- $\tilde{\mathcal{O}}_n\tilde{\mathcal{O}}_n^{\dagger_3} = O_n^2 - \mathfrak{p}O_{n+1}^2 + 2\,(O_nO_{n+3} - O_{n+1}O_{n+2})\,J\varepsilon$.

Lemma 1. *The relationship between $\tilde{\mathcal{U}}_n = \tilde{\mathcal{W}}_n\left(1, 1; 1, \frac{1}{4}\right)$ and $\tilde{\mathcal{O}}_n$ can be given as follows:*

$$\tilde{\mathcal{O}}_n = \frac{1}{2}\tilde{\mathcal{U}}_{n-1}.$$

Theorem 3. *Several facts related to the $\mathcal{DGC}$ Oresme number of type-1 are:*

1. $\lim_{n\to\infty}\tilde{\mathcal{O}}_n = 0$,
2. $\lim_{n\to-\infty}\tilde{\mathcal{O}}_n = -\infty$,
3. $\lim_{n\to\infty}\dfrac{\tilde{\mathcal{O}}_n}{O_n} = 1 + \frac{1}{2}J + \frac{1}{4}\varepsilon + \frac{1}{8}J\varepsilon$,
4. $\lim_{n\to\infty}\dfrac{\tilde{\mathcal{O}}_n}{O_{n-1}} = \frac{1}{2} + \frac{1}{4}J + \frac{1}{8}\varepsilon + \frac{1}{16}J\varepsilon$,
5. $\tilde{\mathcal{O}}_n = \frac{1}{2^n}\left(n + \frac{n+1}{2}J + \frac{n+2}{2^2}\varepsilon + \frac{n+3}{2^3}J\varepsilon\right),\ n \in \mathbb{Z}$
6. $\dfrac{\tilde{\mathcal{O}}_{-n}}{O_n} = -2^{2n}\left(1 + \frac{n-1}{2n}J + \frac{n-2}{4n}\varepsilon + \frac{n-3}{8n}J\varepsilon\right)$.

Proof:
Using the equation (5) and item 5 of Proposition 1, we can make the following computation for the proof of item 4 as follows:

$$\begin{aligned}\lim_{n\to\infty}\frac{\tilde{\mathcal{O}}_n}{O_{n-1}} &= \lim_{n\to\infty}\frac{O_n+O_{n+1}J+O_{n+2}\varepsilon+O_{n+3}J\varepsilon}{O_{n-1}}\\ &= \lim_{n\to\infty}\left(\frac{n}{2(n-1)}+\frac{n+1}{4(n-1)}J+\frac{n+2}{8(n-1)}\varepsilon+\frac{n+3}{16(n-1)}J\varepsilon\right)\\ &= \frac{1}{2}+\frac{1}{4}J+\frac{1}{8}\varepsilon+\frac{1}{16}J\varepsilon.\end{aligned}$$

This indicates validity of item 4. The other items can be proved similarly. □

Theorem 4. *The following linear recurrence relations and summation properties are valid for $\mathcal{DGC}$ Oresme numbers of type-1:*

1. $\tilde{\mathcal{O}}_{n+2}-\frac{3}{4}\tilde{\mathcal{O}}_n+\frac{1}{4}\tilde{\mathcal{O}}_{n-1}=0,$
2. $\tilde{\mathcal{O}}_{n+2}-\frac{3}{4}\tilde{\mathcal{O}}_{n+1}+\frac{1}{16}\tilde{\mathcal{O}}_{n-1}=0,$
3. $\sum_{j=0}^{n-1}\tilde{\mathcal{O}}_j=4\left(\tilde{\mathcal{O}}_1-\tilde{\mathcal{O}}_{n+1}\right),$
4. $\sum_{j=0}^{\infty}\tilde{\mathcal{O}}_j=4\tilde{\mathcal{O}}_1,$
5. $\sum_{j=0}^{n-1}(-1)^j\tilde{\mathcal{O}}_j=\sum_{j=0}^{n-1}(-1)^jO_j(1-J+\varepsilon-J\varepsilon)+(-1)^nO_n(-J+\varepsilon-J\varepsilon)+(-1)^{n+1}O_{n+1}(\varepsilon-J\varepsilon)+(-1)^{n+2}O_{n+2}(-J\varepsilon)+O_1\varepsilon$
6. $\sum_{j=0}^{n-1}\tilde{\mathcal{O}}_{2j}=\left(\sum_{j=0}^{n-1}O_{2j}+\sum_{j=0}^{n-1}O_{2j+1}J\right)(1+\varepsilon)+O_{2n}\varepsilon+\left(O_{2n+1}-O_1\right)J\varepsilon,$
7. $\sum_{j=0}^{n-1}\tilde{\mathcal{O}}_{2j+1}=\left(\sum_{j=0}^{n-1}O_{2j+1}+\sum_{j=0}^{n-1}O_{2j}J\right)(1+\varepsilon)+O_{2n}J+\left(O_{2n+1}-O_1\right)\varepsilon+\left(O_{2n}+O_{2n+2}-O_2\right)J\varepsilon.$

Proof: The proofs can efficiently be conducted by using Proposition 1. For item 6 we have:

$$\begin{aligned}\sum_{j=0}^{n-1}\tilde{\mathcal{O}}_{2j} &= \sum_{j=0}^{n-1}O_{2j}+\sum_{j=0}^{n-1}O_{2j+1}J+\sum_{j=0}^{n-1}O_{2j+2}\varepsilon+\sum_{j=0}^{n-1}O_{2j+3}J\varepsilon\\ &= \left(\sum_{j=0}^{n-1}O_{2j}+\sum_{j=0}^{n-1}O_{2j+1}J\right)(1+\varepsilon)+O_{2n}\varepsilon+\left(O_{2n+1}-O_1\right)J\varepsilon.\end{aligned}$$

□

4 Doubling Approach to $\mathcal{DGC}$ Oresme Numbers

This section, introduces $\mathcal{DGC}$ Oresme numbers of type-2 by using the dual number theory over the generalized complex Oresme numbers.
For this purpose, firstly, we define generalized complex Oresme numbers:

Definition 5. *The nth generalized complex Oresme numbers are defined by:*

$$\hat{\mathcal{O}}_n=O_n+O_{n+1}J, \tag{6}$$

where $J^2=\mathfrak{p}$. The generalized complex Oresme sequence satisfies the following relation:

$$\hat{\mathcal{O}}_{n+2}=\hat{\mathcal{O}}_{n+1}-\frac{1}{4}\hat{\mathcal{O}}_n,\ \forall n\in\mathbb{Z}.$$

The followings are several values of $\hat{\mathcal{O}}_n$:

$$\begin{aligned}\hat{\mathcal{O}}_{-4} &= -64-24J, & \hat{\mathcal{O}}_{-1} &= -2, & \hat{\mathcal{O}}_2 &= \frac{1}{2}+\frac{3}{8}J,\\ \hat{\mathcal{O}}_{-3} &= -24-8J, & \hat{\mathcal{O}}_0 &= \frac{1}{2}J, & \hat{\mathcal{O}}_3 &= \frac{3}{8}+\frac{1}{4}J,\\ \hat{\mathcal{O}}_{-2} &= -8-2J, & \hat{\mathcal{O}}_1 &= \frac{1}{2}+\frac{1}{2}J, & \hat{\mathcal{O}}_4 &= \frac{1}{4}+\frac{5}{32}J.\end{aligned}$$

The generalized complex Oresme sequence is analog to complex Oresme sequence for $\mathfrak{p}=-1$, hyperbolic Oresme sequence for $\mathfrak{p}=1$ and dual Oresme sequence for $\mathfrak{p}=0$.
We now present an alternative $\mathcal{DGC}$ Oresme sequence and its properties.

Definition 6. *The nth $\mathcal{DGC}$ Oresme numbers of type-2 are defined by:*

$$\mathsf{O}_n = \hat{O}_n + \hat{O}_{n+1}\varepsilon, \tag{7}$$

where $\hat{O}_n = O_n + O_{n+1}J$ is generalized complex Oresme number. The $\mathcal{DGC}$ Oresme numbers of type-2 satisfy the following second-order relation

$$\mathsf{O}_{n+2} = \mathsf{O}_{n+1} - \frac{1}{4}\mathsf{O}_n, \; \forall n \in \mathbb{Z}.$$

The followings are several values O_n:

$$\begin{aligned}
\mathsf{O}_{-3} &= -24 - 8J - 8\varepsilon - 2J\varepsilon & \mathsf{O}_1 &= \frac{1}{2} + \frac{1}{2}J + \frac{1}{2}\varepsilon + \frac{3}{8}J\varepsilon, \\
\mathsf{O}_{-2} &= -8 - 2J - 2\varepsilon & \mathsf{O}_2 &= \frac{1}{2} + \frac{3}{8}J + \frac{3}{8}\varepsilon + \frac{1}{4}J\varepsilon, \\
\mathsf{O}_{-1} &= -2 + \frac{1}{2}J\varepsilon & \mathsf{O}_3 &= \frac{3}{8} + \frac{1}{4}J + \frac{1}{4}\varepsilon + \frac{5}{32}J\varepsilon, \\
\mathsf{O}_0 &= \frac{1}{2}J + \frac{1}{2}\varepsilon + \frac{1}{2}J\varepsilon, & \mathsf{O}_4 &= \frac{1}{4} + \frac{5}{32}J + \frac{5}{32}\varepsilon + \frac{3}{32}J\varepsilon.
\end{aligned}$$

Equation (7) formulate another key concept of this paper. Hence, it is natural to seek its algebraic structure and relations. For any O_n and O_m, the standard algebraic operations are defined as follows:

- O_n is equal to O_m if and only if $\hat{O}_n = \hat{O}_m \wedge \hat{O}_{n+1} = \hat{O}_{m+1}$.
- Addition (and hence subtraction) acts component-wise, i.e., $\mathsf{O}_n + \mathsf{O}_m = \left(\hat{O}_n + \hat{O}_m J\right) + \left(\hat{O}_{n+1} + \hat{O}_{m+1}J\right)\varepsilon$.
- The scalar multiplication is calculated as: $\lambda\mathsf{O}_n = (\lambda\hat{O}_n) + (\lambda\hat{O}_{n+1})J$, where $\lambda \in \mathbb{R}$.
- The product of two $\mathcal{DGC}$ Oresme numbers of type-2: $\mathsf{O}_n\mathsf{O}_m = \hat{O}_n\hat{O}_m + \left(\hat{O}_n\hat{O}_{m+1} + \hat{O}_{n+1}\hat{O}_m\right)\varepsilon$.
- The different conjugations* and modules for these numbers can be defined in Table 3.

Conjugates	Modules
$\mathsf{O}_n^{\dagger_1} = \overline{\hat{O}}_n + \overline{\hat{O}}_{n+1}\varepsilon$	$\mathsf{N}_{\mathsf{O}_n}^{\dagger_1} = \mathsf{O}_n\mathsf{O}_n^{\dagger_1}$
$\mathsf{O}_n^{\dagger_2} = \hat{O}_n - \hat{O}_{n+1}\varepsilon$	$\mathsf{N}_{\mathsf{O}_n}^{\dagger_2} = \mathsf{O}_n\mathsf{O}_n^{\dagger_2}$
$\mathsf{O}_n^{\dagger_3} = \overline{\hat{O}}_n - \overline{\hat{O}}_{n+1}\varepsilon$	$\mathsf{N}_{\mathsf{O}_n}^{\dagger_3} = \mathsf{O}_n\mathsf{O}_n^{\dagger_3}$

Table 3 Conjugations and modules of $\mathcal{DGC}$ Oresme numbers of type-2

The following elementary properties are an extension of the familiar relations of Oresme numbers to $\mathcal{DGC}$ versions of type-2. We omit the proofs because they are quite easy to verify using Definition 6, Table 3 and Proposition 1:

Proposition 3. *Let O_n be $\mathcal{DGC}$ Oresme number of type-2. Then, the below properties can be given:*

- $\mathsf{O}_n + \mathsf{O}_n^{\dagger_1} = 2\left(O_n + O_{n+1}\varepsilon\right) = \frac{1}{2^n}(2n + (n+1)\varepsilon)$,
- $\mathsf{O}_n\mathsf{O}_n^{\dagger_1} = O_n^2 - \mathfrak{p}O_{n+1}^2 + 2O_{n+1}\left(O_n - \mathfrak{p}O_{n+2}\right)\varepsilon$,
- $\mathsf{O}_n + \mathsf{O}_n^{\dagger_2} = 2\left(O_n + O_{n+1}J\right) = \frac{1}{2^n}(2n + (n+1)J)$,
- $\mathsf{O}_n\mathsf{O}_n^{\dagger_2} = O_n^2 + \mathfrak{p}O_{n+1}^2 + 2O_nO_{n+1}J$,
- $\mathsf{O}_n + \mathsf{O}_n^{\dagger_3} = 2\left(O_n + O_{n+2}J\varepsilon\right) = \frac{1}{2^n}\left(2n + \frac{n+2}{2}J\varepsilon\right)$,
- $\mathsf{O}_n\mathsf{O}_n^{\dagger_3} = O_n^2 - \mathfrak{p}O_{n+1}^2 + 2\left(O_nO_{n+2} - O_{n+1}^2\right)J\varepsilon$.

Theorem 5. *Several facts related to the $\mathcal{DGC}$ Oresme number of type-2 are:*

1. $\lim_{n\to\infty}\mathsf{O}_n = 0$,
2. $\lim_{n\to-\infty}\mathsf{O}_n = -\infty$,
3. $\lim_{n\to\infty}\dfrac{\mathsf{O}_n}{O_n} = 1 + \frac{1}{2}J + \frac{1}{2}\varepsilon + \frac{1}{4}J\varepsilon$,
4. $\lim_{n\to\infty}\dfrac{\mathsf{O}_n}{O_{n-1}} = \frac{1}{2} + \frac{1}{4}J + \frac{1}{4}\varepsilon + \frac{1}{8}J\varepsilon$,
5. $\mathsf{O}_n = \frac{1}{2^n}\left(n + \frac{n+1}{2}J + \frac{n+1}{2}\varepsilon + \frac{n+2}{2^2}J\varepsilon\right),\; n \in \mathbb{Z}$,
6. $\dfrac{\mathsf{O}_{-n}}{O_n} = -2^{2n}\left(1 + \frac{n-1}{2n}J + \frac{n-1}{2n}\varepsilon + \frac{n-2}{4n}J\varepsilon\right)$.

The overline represents the standard generalized complex conjugate.

Proof: Using equation (7) and item 7 of Proposition 1, we can make the following computation for the proof of item 6 as follows:

$$\begin{aligned}\frac{\mathsf{O}_{-n}}{O_n} &= \frac{\hat{O}_{-n}+\hat{O}_{-n+1}\varepsilon}{O_n} \\ &= \frac{O_{-n}+O_{-n+1}J+O_{-n+1}\varepsilon+O_{-n+2}J\varepsilon}{O_n} \\ &= \frac{-n2^{n}-(n-1)2^{n-1}J-(n-1)2^{n-1}\varepsilon-(n-2)2^{n-2}J\varepsilon}{\frac{n}{2^n}} \\ &= -2^{2n}\left(1+\frac{n-1}{2n}J+\frac{n-1}{2n}\varepsilon+\frac{n-2}{4n}J\varepsilon\right).\end{aligned}$$

The proof of item 6 is verified. The other items can be seen similarly.

□

Theorem 6. *The following linear recurrence relations and summation properties are valid for $\mathcal{DGC}$ Oresme numbers of type-2:*

1. $\mathsf{O}_{n+2}-\frac{3}{4}\mathsf{O}_n+\frac{1}{4}\mathsf{O}_{n-1}=0$,
2. $\mathsf{O}_{n+2}-\frac{3}{4}\mathsf{O}_{n+1}+\frac{1}{16}\mathsf{O}_{n-1}=0$,
3. $\sum_{j=0}^{n-1}\mathsf{O}_j=4\left(\mathsf{O}_1-\mathsf{O}_{n+1}\right)$,
4. $\sum_{j=0}^{\infty}\mathsf{O}_j=4\mathsf{O}_1$,
5. $\sum_{j=0}^{n-1}(-1)^j\mathsf{O}_j=\sum_{j=0}^{n-1}(-1)^jO_j(1-J-\varepsilon+J\varepsilon)+(-1)^nO_n(-J-\varepsilon+J\varepsilon)+(-1)^{n+1}O_{n+1}J\varepsilon+O_1J\varepsilon$,
6. $\sum_{j=0}^{n-1}\mathsf{O}_{2j}=\sum_{j=0}^{n-1}O_{2j}\left(1+J\varepsilon\right)+\sum_{j=0}^{n-1}O_{2j+1}\left(J+\varepsilon\right)+O_{2n}J\varepsilon$,
7. $\sum_{j=0}^{n-1}\mathsf{O}_{2j+1}=\sum_{j=0}^{n-1}O_{2j+1}\left(1+J\varepsilon\right)+(\sum_{j=0}^{n-1}O_{2j}+O_{2n})\left(J+\varepsilon\right)+\left(O_{2n+1}-O_1\right)J\varepsilon$.

Proof: According to the fundamental properties in Proposition 1, the proof is relatively easy to verify. For item 7 we obtain:

$$\begin{aligned}\sum_{j=0}^{n-1}\mathsf{O}_{2j+1} &= \sum_{j=0}^{n-1}\hat{O}_{2j+1}+\sum_{j=0}^{n-1}\hat{O}_{2j+2}\varepsilon \\ &= \sum_{j=0}^{n-1}O_{2j+1}+\sum_{j=0}^{n-1}O_{2j+2}J+\sum_{j=0}^{n-1}O_{2j+2}\varepsilon+\sum_{j=0}^{n-1}O_{2j+3}J\varepsilon \\ &= \sum_{j=0}^{n-1}O_{2j+1}\left(1+J\varepsilon\right)+(\sum_{j=0}^{n-1}O_{2j}+O_{2n})\left(J+\varepsilon\right)+\left(O_{2n+1}-O_1\right)J\varepsilon.\end{aligned}$$

□

5 Conclusion

The main target of this study is to obtain $\mathcal{DGC}$ Oresme numbers of type-1 and type-2 by introducing their general recurrence relations for any real number $\mathfrak{p}$ in the light of the study [45]. The striking part of this paper is that one can reduce the calculations to dual complex, hyper-dual and dual-hyperbolic Oresme for real values $\mathfrak{p}=-1$, $\mathfrak{p}=0$ and $\mathfrak{p}=1$, respectively. Considering these values, the above mentioned special Oresme numbers are generalized from the viewpoint of definition, algebraic properties, recurrence relations and well-known identities in Sections 3 and 4. Hence, Sections 3 and 4 are directly linked to the paper dual-complex case for $\mathfrak{p}=-1$, the hyper-dual case for $\mathfrak{p}=0$, and the dual-hyperbolic case for $\mathfrak{p}=1$ in view of Oresme. These classifications can be seen in Table 4.

Cases	type-1	type-2 (doubling)	Condition ($\varepsilon^2=0$)
Dual-complex	$O_n+O_{n+1}i+O_{n+2}\varepsilon+O_{n+3}i\varepsilon$	$O_n+O_{n+1}i+O_{n+1}\varepsilon+O_{n+2}i\varepsilon$	$i^2=-1$
Hyper-dual	$O_n+O_{n+1}\epsilon+O_{n+2}\varepsilon+O_{n+3}\epsilon\varepsilon$	$O_n+O_{n+1}\epsilon+O_{n+1}\varepsilon+O_{n+2}\epsilon\varepsilon$	$\epsilon^2=0$, $(\varepsilon\neq 0,\epsilon\neq 0,\varepsilon\epsilon\neq 0)$
Dual-hyperbolic	$O_n+O_{n+1}j+O_{n+2}\varepsilon+O_{n+3}j\varepsilon$	$O_n+O_{n+1}j+O_{n+1}\varepsilon+O_{n+2}j\varepsilon$	$j^2=1$, $(j\neq\pm 1)$

Table 4 $\mathcal{DGC}$ Oresme numbers of type-1 and type-2 for $\mathfrak{p}\in\{-1,0,1\}$

With a similar thought, our next goal is to examine multiplicative identities include $\mathcal{DGC}$ Oresme numbers. Additionally, we are planned to construct a bridge between $\mathcal{DGC}$ Oresme numbers and quaternions. Further, the construction of the same relation is planned to carry out for $\mathcal{DGC}$ k-Oresme numbers after examining the necessary properties.

6 References

1 A. F. Horadam, *Generating functions for powers of a certain generalised sequence of numbers*, Duke Math. J. , **32** (3) (1965), 437-446.

2 A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*, Fibonacci Quart., **3** (3) (1965), 161-176.
3 A. F. Horadam, *Special properties of the sequence $W_n(a, b; p, q)$*, Fibonacci Quart., **5** (1967), 424-434.
4 P. Haukkanen, *A note on Horadam's sequence*, Fibonacci Quart., **40** (4) (2002), 358-361.
5 A. F. Horadam, *A generalized Fibonacci sequence*, Amer. Math. Monthly, **68** (5) (1961), 455-459.
6 I. Mezo, *Several generating functions for second-order recurrence sequences*, J. Integer Seq, **12** (2009), Article 09.3.7.
7 E. Kılıç, E. Tan, *More general identities involving the terms of $\{W_n(a, b; p, q)\}$*, Ars Combin., **93** (2009), 459-461.
8 H. Belbachir, A. Belkhir, *On some generalizations of Horadam's numbers*, Filomat, **32** (14) (2018), 5037-5052.
9 N. Taşkara, K. Uslu, Y. Yazlık, N. Yılmaz, *The construction of Horadam numbers in terms of the determinant of tridiagonal matrices*, AIP Conf. Proc., 1389, **1** (2011), 367-370.
10 Y. Yazlık, N. Taskara, *A note on generalized k-Horadam sequence*, Comput. Math. Appl., **63** (1) (2012), 36-41.
11 N. J. A. Sloane, *The online encyclopedia of integer sequences*, (1964), Available at http://oeis.org/.
12 A. F. Horadam, *Oresme numbers*, Fibonacci Quart., **12** (3) (1974), 267-271.
13 L. Hogben, *Mathematics in the making*, Macdonald, 1960.
14 C. K. Cook, *Some sums related to sums of Oresme numbers*, In Applications of Fibonacci Numbers, Springer, Dordrecht, 2004, pp. 87-99.
15 T. Goy, R. Zatorsky, *On Oresme numbers and their connection with Fibonacci and Pell numbers*, Fibonacci Quart., **57** (3) (2019), 238-245.
16 G. Cerda-Morales, *Oresme Polynomials and their derivatives*, arXiv preprint arXiv:1904.01165, 2019.
17 F. R. V. Alves, R. P. M. Vieira, P. M. M. C. Catarino, *Visualizing the Newtons fractal from the recurring linear sequence with Google Colab: an example of Brazil X Portugal research*, International Electric Journal of Mathematics Education, **15** (3) (2020), em0594.
18 M. C. dos Santos Mangueira, R. P. M. Vieira, F. R. V. Alves, P. M. M. C. Catarino, *The Oresme sequence: The generalization of its matrix form and its hybridization process*, NNTDM, **27** (1) (2021), 101-111.
19 I. M. Yaglom, *Complex numbers in geometry*, Academic Press, New York. 1968.
20 I. M. Yaglom, *A simple non-Euclidean geometry and its physical basis*, Springer-Verlag, New York. 1979.
21 P. Fjelstad, *Extending special relativity via the perplex numbers*, Amer. J. Phys, **54** (5) (1986), 416–422.
22 G. Sobczyk, *The hyperbolic number plane*, College Math. J., **26** (4) (1995), 268-280.
23 E. Pennestrì, R. Stefanelli, *Linear algebra and numerical algorithms using dual numbers*, Multibody Syst. Dyn., **18** (2007), 323-344.
24 E. Study, *Geometrie der dynamen: Die zusammensetzung von kräften und verwandte gegenstände der geometrie bearb.* Leipzig, B.G. Teubner. 1903.
25 I. Kantor, A. Solodovnikov, *Hypercomplex Numbers*, Springer-Verlag, New York, 1989.
26 A. A. Harkin, J. B. Harkin, *Geometry of Generalized Complex Numbers*, Math. Mag., **77** (2) (2004), 118-129.
27 V. Majernik, *Multicomponent Number Systems*, Acta Phys. Pol. A, **90** (3) (1996), 491-498.
28 M. Akar, S. Yüce, S. Şahin, *On the dual hyperbolic numbers and the complex hyperbolic numbers*, Journal of Computer Science and Computational Mathematics, **8** (1) (2018), 1-6.
29 J. Cockle, *On a New Imaginary in Algebra*, Philosophical magazine, London-Dublin-Edinburgh, **34:226** (1849) , 37-47.
30 D. Alfsmann , *On families of 2N-dimensional hypercomplex algebras suitable for digital signal processing*, in Proc. European Signal Processing Conf. (EUSIPCO), Florence, Italy, 2006.
31 H. H. Cheng, S. Thompson, *Dual polynomials and complex dual numbers for analysis of spatial mechanisms*, Proc. of ASME 24th Biennial Mechanisms Conference, Irvine, CA, 19-22 August 1996.
32 H. H. Cheng, S. Thompson, *Singularity analysis of spatial mechanisms using dual polynomials and complex dual numbers*, ASME. J. Mech. Des., **121** (2) (1999), 200-205.
33 F. Messelmi, *Dual-complex numbers and their holomorphic functions*. hal-01114178. 2015.
34 D. Rochon, M. Shapiro, *On algebraic properties of bicomplex and hyperbolic numbers*, An. Univ. Oradea Fasc. Mat., **11** (2004), 71-110.
35 H. Toyoshima, *Computationally efficient bicomplex multipliers for digital signal processing*, IEICE Trans Inf Syst., **E81-D** (2) (1989), 236-238.
36 C. Segre, *Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici (The real representation of complex elements and hyperalgebraic entities)*, Math. Ann., **40** (1892), 413-467.
37 F. Catoni, D. Boccaletti, R. Cannata, V. Catoni, E. Nichelatti, P. Zampetti, *The mathematics of Minkowski space-time with an introduction to commutative hypercomplex numbers*, Birkhauser Verlag, Basel, Boston, Berlin, 2008.
38 S. Olariu, *Complex numbers in n dimensions*, North-Holland Mathematics Studies, Elsevier, Amsterdam, Boston, 2002.
39 M. Bilgin, S. Ersoy, *Algebraic properties of bihyperbolic numbers*, Adv. Appl. Clifford Algebr., **30** (2020), Article no. 13.
40 A. A. Pogorui, R. M. Rodriguez-Dagnino, R. D. Rodrigue-Said, *On the set of zeros of bihyperbolic polynomials*, Complex Var. Elliptic Equ., **53** (7) (2008), 685-690.
41 F. Catoni, R. Cannata, V. Catoni, P. Zampetti, *N-dimensional geometries generated by hypercomplex numbers*, Adv. Appl. Clifford Algebr., **15** (2005), 1-25.
42 A. Cohen, M. Shoham, *Principle of transference-An extension to hyper-dual numbers*, Mech. Mach. Theory, **125** (2018), 101-110.
43 J. A. Fike, S. Jongsma, J. J. Alonso, E. Van Der. Weide, *Optimization with gradient and hessian information calculated using hyper-dual numbers*, 29th AIAA Applied Aerodynamics Conference, 27-30 June 2011, Honolulu, Hawaii.
44 J. A Fike, J. J. Alonso, *Automatic differentiation through the use of hyper-dual numbers for second derivatives*. Lecture Notes in Computational Science and Engineering book series (LNCSE), **87** (2011), 163-173.
45 N. Gürses, G. Y. Şentürk, S. Yüce, *A study on dual-generalized complex and hyperbolic-generalized complex Numbers*, GUJS, **34**(1) (2021), 180-194.
46 S. Köme, C. Köme, Y. Yazlik, *Dual-complex generalized k-Horadam numbers*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **70** (1) (2021), 117-129.
47 E. Tan, N. R. Ait-Amrane, I. Gök, *Hyper-Dual Horadam quaternions*, Preprints, (2021), 2021030435.
48 T. D. Şentürk, G. Bilgici, A. Daşdemir, Z. Ünal, *A study on Horadam hybrid numbers*, Turk J Math **44** (4) (2020), 1212-1221.
49 A. Szynal-Liana, *The Horadam hybrid numbers*, Discuss. Math. Gen. Algebra Appl., **38** (2018), 91-98.
50 S. Halici, *On bicomplex Fibonacci numbers and their generalization*, In: Flaut C., Hošková-Mayerová Š., Flaut D. (eds) Models and Theories in Social Systems. Studies in Systems, Decision and Control, Springer, **179** (2019).
51 D. Bród, A. Szynal-Liana, I. Włoch, *Two generalizations of dual-hyperbolic balancing numbers*, Symmetry, **12** (11) (2020), 1866.
52 G. Y. Şentürk, N. Gürses, S. Yüce, *Fundamental properties of extended Horadam numbers*, Number Theory and Discrete Mathematics, (2021), (in press).

Conference Proceeding Science and Technology, 4(2), 2021, 215–225

18th International Geometry Symposium, 2021.

On Semiconformal Curvature Tensor in (k,μ)-Contact Metric Manifold

ISSN: 2651-544X
http://cpostjournal.org/

Jay Prakash Singh[1,*] *Mohan Khatri*[2]

[1] *Department of Mathematics and Computer Science, Mizoram University, Aizawl-796004, India, ORCID:0000-0002-2050-3524*
[2] *Department of Mathematics and Computer Science, Mizoram University, Aizawl-796004, India, ORCID:0000-0002-7867-7148*
** Corresponding Author E-mail: jpsmaths@gmail.com*

Abstract: The objective of the present paper is to study the (k,μ)-contact metric manifold with the semiconformal curvature tensor. The (k,μ)-contact metric manifold satisfying $P\cdot R=0$ and semiconformally flat are studied and the conditions under which it is η-Einstein manifold are established. Further, $P\cdot S=0$ is investigated and the relation for Ricci tensor is obtained. Also, some results for η-Einstein (k,μ)-contact metric manifold satisfying the condition $P\cdot S=0$ are established. Finally, h-semiconformally semi-symmetric (k,μ)-contact metric manifold and ϕ-semiconformally semi-symmetric (k,μ)-contact metric manifold are introduced and shown that non-Sasakian h-semiconformally semi-symmetric (k,μ)-contact metric manifold and non-Sasakian ϕ-semiconformally semi-symmetric (k,μ)-contact metric manifold are η-Einstein manifold if $\mu\neq 1$ and $\mu\neq\frac{n-1}{n}$ respectively.

Keywords: η-Einstein manifold, h-semiconformally semi-symmetric, (k,μ)-contact metric manifold, ϕ-semiconformally semi-symmetric, semiconformal curvature.

1 Introduction

In 2016, Kim [1, 2] introduced a new type of curvature tensor, named as semiconformal curvature tensor, which remain invariant under conharmonic transformation. It is observed that conformal curvature tensor and conharmonic curvature tensor are special cases of semiconformal curvature tensor. Later, Kim [2] studied the geometrical properties of semiconformal curvature tensor on pseudo semiconformally symmetric manifold. The semiconformal curvature tensor P of type $(1,3)$, as defined by Kim, on a Riemannian manifold $(M^{2n+1},g),(n>1)$ is as follows:

$$P(U,V)W=-(2n-1)b\,C(U,V)W+[a+(2n-1)b]\,H(U,V)W, \tag{1}$$

where, a,b are constants and not simultaneously zero, $C(U,V)W$ denotes the conformal curvature tensor of type $(1,3)$ whereas, $H(U,V)W$ denotes the conharmonic curvature tensor of type $(1,3)$. The expression of conformal curvature tensor of type $(1,3)$ and the conharmonic curvature tensor of type $(1,3)$ are given as:

$$\begin{aligned} C(U,V)W &= R(U,V)W-\frac{1}{(2n-1)}\Big[S(V,W)U-S(U,W)V+g(V,W)QU \\ &- g(U,W)QV\Big]+\frac{r}{2n(2n-1)}\Big[g(V,W)U-g(U,W)V\Big], \end{aligned} \tag{2}$$

and

$$\begin{aligned} H(U,V)W &= R(U,V)W-\frac{1}{(2n-1)}\Big[S(V,W)U-S(U,W)V+g(V,W)QU \\ &- g(U,W)QV\Big], \end{aligned} \tag{3}$$

where, r and R are the scalar curvature and Riemannian curvature of type $(1,3)$ of the manifold M^{2n+1} respectively and S is the Ricci tensor of the manifold, given by $g(QU,V)=S(U,V)$, where Q is the Ricci-operator. In consequence of(1),(2) and (3), the semiconformal curvature tensor $\tilde{P}$ of type (0,4) assume the following form:

$$\begin{aligned} \tilde{P}(U,V,W,X) &= a\tilde{R}(U,V,W,X)-\frac{a}{(2n-1)}\Big[S(V,W)g(U,X) \\ &- S(U,W)g(V,X)+S(U,X)g(V,W)-S(V,X)g(U,W)\Big] \\ &- \frac{br}{2n}\Big[g(V,W)g(U,X)-g(U,W)g(V,X)\Big], \end{aligned} \tag{4}$$

where, $\tilde{P}(U,V,W,X)=g(P(U,V)W,X)$ and $\tilde{R}(U,V,W,X)=g(R(U,V)W,X)$.

For $a = 1$ and $b = -\frac{1}{(2n-1)}$, the semiconformal curvature becomes conformal curvature tensor and for $a = 1$ and $b = 0$, the semiconformal curvature tensor reduces to conharmonic curvature tensor. Recently, De and Suh [3] studied the geometrical properties of weakly semiconformally symmetric manifolds. In [4], almost pseudo semiconformally symmetric manifolds is studied.

The k-nullity distribution on a contact metric manifold $(M; \phi, \xi, \eta, g)$ for any $p \in T_p(M)$ and for real numbers k is as follows:

$$N(k) : p \to N_p(k) = [W \in T_p(M) : R(U, V)W = k\{g(V, W)U - g(U, W)V\}], \tag{5}$$

for any $U, V, W \in T_p(M)$, where $T_p(M)$ denotes the tangent vector space at any point p on the manifold M. The notion of k-nullity distribution on a contact metric manifold was introduced by Tanno [5]. Blair et al. [6] introduced (k, μ)-nullity distribution which is the generalized notion of k-nullity distribution and defined it as, "A (k, μ)-nullity distribution on a contact metric manifold $(M; \phi, \xi, \eta, g)$ for any $p \in T_p(M)$ and for real numbers k, μ is as follows:

$$\begin{aligned} N_p(k, \mu) = \{W \in T_p(M) : R(U, V)W = k[g(V, W)U - g(U, W)V] \\ +\mu[g(hV, W)U - g(hU, W)V]\}, \end{aligned} \tag{6}$$

where h is a tensor field of type (1,3) defined by $h = \frac{1}{2} L_\xi \phi$, where L denotes the Lie differentiation. If the characteristic vector field, i.e.,ξ of a contact metric manifold M^{2n+1} lies in (k, μ)-nullity distribution, then it is called (k, μ)-contact metric manifold."A full classification of (k, μ)-contact metric manifolds was given by Boeckx [7]. This class contains Sasakian manifolds for $k = 1$. The k-nullity distribution is a subclass of (k, μ)-nullity distribution.

The notion of symmetric spaces was introduced by Cartan [8]. According to him, "An n-dimensional Riemannian manifold is said to be locally symmetric if its curvature tensor R satisfies $\nabla R = 0$, where '∇', represent covariant differentiation with respect to metric tensor". Cartan noticed that all locally symmetric and 2-dimensional Riemannian spaces belong to a class of Riemannian manifold satisfying the condition $R(U, V) \cdot R = 0$. Kowalski [9] studied 3-dimensional Riemannian manifold satisfying $R(U, V) \cdot R = 0$. In recent years, many geometers [10–15] studied (k, μ)-contact metric manifolds, as it is an interesting topic in differential geometry. One of its interesting characteristics is that there is a special case of (k, μ)-spaces which were the first known example of a non-sasakian locally ϕ-symmetric space [16]. Sarkar and De [17] studied on the (k, μ)-contact metric manifold and obtained some results based on quasi-conformal curvature tensor. Motivated by the above studies, we studied semiconformal curvature tensor in (k, μ)-contact metric manifold.

The present paper is organized as follows: After preliminaries, in Section 3, we studied the flatness of semiconformal curvature tensor in (k, μ)-contact metric manifold and observed that under such condition it is η-Einstein manifold. In section 4, the manifold satisfying $P \cdot S = 0$ is studied and observed that the Ricci tensor satisfies the relation (33). We showed that for η-Einstein (k, μ)-contact metric manifold with semiconformally Ricci semi-symmetric the relation $2a(n^2 + nk - 1) = (2n - 1)(br - ak)$ holds. Next, in Section 5 and 6, we introduced h-semiconformally semi-symmetric (k, μ)-contact metric manifold and ϕ-semiconformally semi-symmetric (k, μ)-contact metric manifold respectively. In both cases, under certain conditions of μ, (k, μ)-contact metric manifold is η-Einstein manifold. Also based on previous results, the expression of Ricci tensor for different semi-symmetric conditions is obtained and tabulated. Finally, in Section 7, we investigated semiconformally semi-symmetric (k, μ)-contact metric manifold. Throughout this paper, we assumed that $a \neq 0$.

2 Preliminaries

Here, we listed out some of the basic results obtained by different authors which we used in the study of (k, μ)-contact metric manifold. A contact manifold [18] is a $(2n + 1)$-dimensional differentiable manifold M^{2n+1}, together with a global differentiable 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1}. This 1-form η is called the contact form of M^{2n+1}. A contact manifold satisfies the following relations:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{7}$$

$$g(U, \phi V) = d\eta(U, V), \tag{8}$$

$$g(U, \phi V) = -g(V, \phi U), \quad g(U, \xi) = \eta(U), \tag{9}$$

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{10}$$

for all vector fields U, V on M. Here, ϕ is a tensor field of type (1,1), ξ is a contravariant global vector field called the characteristic vector field of the manifold and g is the Riemannian metric associated with the manifold. The contact metric manifold together with the Riemannian metric g is called contact Riemannian manifold $(M; \phi, \xi, \eta, g)$. The vector field h of type (1,3) is symmetric in contact metric manifold and satisfies:

$$h\phi = -\phi h, \quad Trh = Tr\phi h = 0, \quad h\xi = 0. \tag{11}$$

$$\nabla_U \xi = -\phi U - \phi h U. \tag{12}$$

In (k, μ)-contact metric manifold the following relation holds [6, 19],

$$h^2 = (k - 1)\phi^2, \;\; k \leq 1, \tag{13}$$

$$R(U, V)\xi = k[\eta(V)U - \eta(U)V] + \mu[\eta(V)hU - \eta(U)hV], \tag{14}$$

$$R(\xi, U)V = k[g(U, V)\xi - \eta(V)U] + \mu[g(hU, V)\xi - \eta(V)hU], \tag{15}$$

$$S(U, \xi) = 2nk\eta(U), \tag{16}$$

$$S(\phi U, \phi V) = S(U, V) - 2nk\eta(U)\eta(V) - 2(2n - 2 + \mu)g(hU, V), \tag{17}$$

$$S(U, V) = [2(n-1) - n\mu]g(U, V) + [2(n-1) + \mu]g(hU, V) \\ +[2(1-n) + n(2k + \mu)]\eta(U)\eta(V), \ \ n \geq 1, \tag{18}$$

$$(\nabla_U \eta)(V) = g(U + hU, \phi V), \tag{19}$$

$$(\nabla_U h)(V) = \{(1-k)g(U, \phi V) + g(U, h\phi V)\}\xi \\ +\eta(V)\{h(\phi U + \phi hU)\} - \mu\eta(U)\phi hV. \tag{20}$$

Blair et al. [6] obtained the relation between the Ricci operator Q and ϕ, which is as follows:

Proposition 1. *Let M^{2n+1} be a (k, μ)-contact metric manifold. Then the relation*

$$Q\phi - \phi Q = 2[2(n-1) + \mu]h\phi,$$

holds.

In (k, μ)-contact metric manifold, the Ricci operator Q does not generally commute with ϕ. We recalled the results obtained by Yildiz [20] which is as follows:

Lemma 1. *[20] In a non-Sasakian (k, μ)-contact metric manifold the following conditions are equivalent:*
(i) η-Einstein manifold,
(ii) $Q\phi = \phi Q$.

3 Semiconformally flat (k, μ)-contact metric manifold

The flatness of the semiconformal curvature tensor in (k, μ)-contact metric manifold M^{2n+1} is studied in this section. From (4), the semiconformal curvature tensor of type (1,3) takes the form

$$P(U, V)W = aR(U, V)W - \frac{a}{(2n-1)}\big[S(V, W)U - S(U, W)V \\ +g(V, W)QU - g(U, W)QV\big] - \frac{br}{2n}\big[g(V, W)U - g(U, W)V\big], \tag{21}$$

for all vector fields U, V, W on $T_p(M)$.

Definition 1. *A (k, μ)-contact metric manifold M^{2n+1} is said to be η-Einstein [21] if its Ricci tensor S satisfies*

$$S(U, V) = \alpha g(U, V) + \beta\eta(U)\eta(V), \tag{22}$$

for all vector fields U, V and some real constants α and β. For $\beta = 0$, it reduces to Einstein manifold.

Theorem 1. *If (k, μ)-contact metric manifold $M^{2n+1}(n > 1)$, is semiconformally flat, then it is η-Einstein manifold, provided $\mu \neq 1$.*

Proof: Suppose that $M^{2n+1}(\phi, \xi, \eta, g)$ is a semiconformally flat (k, μ)-contact metric manifold. Then, from (4), we obtain

$$\tilde{R}(U, V, W, X) = \frac{1}{(2n-1)}\Big[S(V, W)g(U, X) - S(U, W)g(V, X) + g(V, W)S(U, X) \\ -g(U, W)S(V, X)\Big] + \frac{br}{2na}\Big[g(V, W)g(U, X) - g(U, W)g(V, X)\Big]. \tag{23}$$

Putting $W = \xi$ in (23) and using (14), (16) we get

$$k\Big[\eta(V)g(U, X) - \eta(U)g(V, X)\Big] + \mu\Big[\eta(V)g(hU, X) - \eta(U)g(hV, X)\Big] \\ = \frac{1}{(2n-1)}\Big[2nk\eta(V)g(U, X) - 2nk\eta(U)g(V, X) + \eta(V)S(U, X) \\ -\eta(U)S(V, X)\Big] + \frac{br}{2na}\Big[\eta(V)g(U, X) - \eta(U)g(V, X)\Big]. \tag{24}$$

Put $U = \xi$ in (24) we obtain

$$S(V, X) = \Big[\frac{-2nak - br(2n-1)}{2na}\Big]g(V, X) \\ + \Big[\frac{2nak(2n+1) + br(2n-1)}{2na}\Big]\eta(X)\eta(V) + \mu(2n-1)g(hV, X). \tag{25}$$

Using (18) and (25), we obtain the following relation

$$S(V,X) = A_1 g(V,X) + B_1\eta(V)\eta(X), \tag{26}$$

where

$$A_1 = \frac{(1-n)[2(2n-1)(\mu+br) - 4nak(\mu-1)] + \mu(2n-1)(2n\mu-k-br)}{4na(\mu-1)(1-n)},$$

and

$$B_1 = \frac{(1-n)[4nak(k-1)(2n+1) - 2(2n-1)(2na\mu+br)]}{4na(\mu-1)(1-n)} + \frac{\mu(2n-1)[br(3-4n) - 2na(n\mu+k)]}{4na(\mu-1)(1-n)}.$$

Thus, M is η-Einstein manifold. □

Combining Lemma 1 and Theorem 1, we can obtain the following:

Corollary 1. *If non-Sasakian (k,μ)-contact metric manifold $M^{2n+1}(n>1)$, is semiconformally flat and $\mu \neq 1$ then $Q\phi = \phi Q$ is satisfied.*

4 A (k,μ)-contact metric manifold M^{2n+1} satisfying $P \cdot S = 0$

Definition 2. *A (k,μ)-contact metric manifold M^{2n+1} is said to be semiconformally Ricci semi-symmetric if the semiconformal curvature tensor satisfies the condition,*

$$P(U,V)\cdot S = 0,$$

for any vector fields U, V on M.

Theorem 2. *If (k,μ)-contact metric manifold M^{2n+1}, $(n>1)$ is semiconformally Ricci semi-symmetric, then the Ricci tensor S of M^{2n+1} satisfies the relation (33).*

Proof: Suppose (k,μ)-contact metric manifold is semiconformally Ricci semi-symmetric, i.e.,

$$P(\xi,V)\cdot S = 0,$$

which implies,

$$S(P(\xi,V)U,W) + S(U,P(\xi,V)W) = 0. \tag{27}$$

Putting $W=\xi$ in (27), we obtain

$$2nk\eta(P(\xi,V)U) + S(U,P(\xi,V)\xi) = 0. \tag{28}$$

From (21), we have

$$P(\xi,V)U = aR(\xi,V)U - \frac{a}{2n-1}\big[S(V,U)\xi - 2nk\eta(U)V + g(V,U)Q\xi - \eta(U)QV\big] - \frac{br}{2n}\big[g(V,U)\xi - \eta(U)V\big]. \tag{29}$$

In (29), taking its inner product with ξ and using (14), we get the following expression

$$\eta(P(\xi,V)U) = \Big[ak\big(1-\frac{2n}{2n-1}\big) - \frac{br}{2n}\Big]g(U,V) + \Big[-ak + \frac{4nak}{2n-1} + \frac{br}{2n}\Big]\eta(U)\eta(V) - \frac{a}{2n-1}S(U,V) + a\mu g(hV,U). \tag{30}$$

Again, using (21) we can have

$$P(\xi,V)\xi = ak\big[\eta(V)\xi - V\big] - a\mu hV - \frac{a}{2n-1}\big[2nk\eta(V)\xi - 2nkV + \eta(V)Q\xi - QV\big] - \frac{br}{2n}\big[\eta(V)\xi - V\big]. \tag{31}$$

In regard of (31), we get

$$S(U,P(\xi,V)\xi) = 2nk\Big[ak - \frac{br}{2n} - \frac{4nak}{2n-1}\Big]\eta(U)\eta(V) + \Big[-ak + \frac{br}{n} + \frac{2nka}{2n-1}\Big]S(U,V) - a\mu S(U,hV) + \frac{a}{2n-1}S^2(U,V). \tag{32}$$

From (28), (30) and (32) we obtain

$$S^2(U,V) = \frac{k}{a}\big[2nak + br(2n-1)\big]g(U,V) + \frac{(2n-1)}{na}\big[nak - br\big]S(U,V)$$
$$-2nk(2n-1)\mu g(hU,V) + \mu(2n-1)S(U,hV). \tag{33}$$

□

Theorem 3. *If M^{2n+1}, $(n>1)$ is a (k,μ)-contact metric manifold whose Ricci tensor S is η-Einstein and satisfies $P \cdot S = 0$, then $2a(n^2 + nk - 1) = (2n-1)(br - ak)$ holds.*

Proof: From (18), we see that (k,μ)-contact metric manifold is η-Einstein if and only if $\mu = -2(n-1)$ is satisfied. For this value, (17) gives

$$S(U,V) = 2(n^2-1)g(U,V) - 2(n^2 - nk - 1)\eta(U)\eta(V). \tag{34}$$

and,

$$QU = 2(n^2-1)U - 2(n^2 - nk - 1)\eta(U)\xi. \tag{35}$$

By using (21), (34) and (35), we get the following relations,

$$S(P(\xi,V)U,\xi) = 2nk\big[ak - \frac{br}{2n} + \frac{2a[n^2+nk-1]}{2n-1}\big]$$
$$\big[g(V,U) - \eta(U)\eta(V)\big] - 4nak(n-1)g(hV,X). \tag{36}$$

and,

$$S(U,P(\xi,V)\xi) = \big[ak - \frac{br}{2n} + \frac{2a[n^2+nk-1]}{2n-1}\big]$$
$$\big[2nk\eta(U)\eta(V) - S(U,V)\big] + 2a(n-1)S(U,hV). \tag{37}$$

Putting $W = \xi$ in (27), we get

$$S(P(\xi,V)U,\xi) + S(U,P(\xi,V)\xi) = 0. \tag{38}$$

Making use of (34), (36), (37) in (38), we obtain

$$2(n^2 - nk - 1)\big[ak - \frac{br}{2n} + \frac{2a[n^2+nk-1]}{2n-1}\big][\eta(U)\eta(V) - g(U,V)]$$
$$+4a(n-1)(n^2 - 1 - nk)g(U,hV) = 0. \tag{39}$$

Contracting (39) with respect to U and V, we get

$$4n(n^2 - nk - 1)\big[ak - \frac{br}{2n} + \frac{2a[n^2+nk-1]}{2n-1}\big] = 0. \tag{40}$$

Clearly, we see that either $n^2 - nk - 1 = 0$ or $2a(n^2 + nk - 1) = (2n-1)(br - ak)$. For the case when $n^2 - nk - 1 = 0$ implies $k = \frac{n^2-1}{n}$, a contradiction as $k \le 1$. □

5 h-semiconformally semi-symmetric non-Sasakian (k,μ)-contact metric manifold

Definition 3. *A Riemannian manifold (M^{2n+1}, g) is said to be h-semiconformally semi-symmetric if*

$$P(U,V) \cdot h = 0,$$

holds for any vector fields U, V on M.

Lemma 2. *[6] Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. Then*

$$R(U,V)hW - hR(U,V)W = \Big\{k\big[g(hV,W)\eta(U) - g(hU,W)\eta(V)\big] +$$
$$\mu(k-1)\big[g(U,W)\eta(V) - g(V,W)\eta(U)\big]\Big\}\xi + k\Big\{g(V,\phi W)\phi hU -$$
$$g(U,\phi W)\phi hV + g(W,\phi hV)\phi U - g(W,\phi hU)\phi V + \eta(W)\big[\eta(U)hV -$$
$$\eta(V)hU\big]\Big\} - \mu\Big\{\eta(V)\big[(1-k)\eta(W)U + \mu\eta(U)hW\big] -$$
$$\eta(U)\big[(1-k)\eta(W)V + \mu\eta(V)hW\big] + 2g(U,\phi V)\phi hW\Big\}, \tag{41}$$

for any vector fields U, V, W on M.

Theorem 4. *If $M^{2n+1}, (n > 1)$ is a non-Sasakian h-semiconformally semi-symmetric (k, μ)-contact metric manifold with $\mu \neq 1$, then M is η-Einstein manifold.*

Proof: Suppose M is h-semiconformally semi-symmetric,i.e.,

$$(P(U,V) \cdot h)W = 0.$$

or,

$$P(U,V)hW - hP(U,V)W = 0. \tag{42}$$

Using (21) in (42), we obtain

$$\begin{aligned} a\big[R(U,V)hW - hR(U,V)W\big] - \frac{a}{2n-1}\big[S(V,hW)U - S(V,W)hU - \\ S(U,hW)V + S(U,W)hV + g(V,hW)QU - g(V,W)hQU - \\ g(U,hW)QV + g(U,W)hQV\big] - \frac{br}{2n}\big[g(V,hW)U - g(V,W)hU - \\ g(U,hW)V + g(U,W)hV\big] = 0. \end{aligned} \tag{43}$$

Using (41) in (43),

$$\begin{aligned} a\Bigg[\Big\{k\big[g(hV,W)\eta(U) - g(hU,W)\eta(V)\big] + \mu(k-1)\big[g(U,W)\eta(V) - \\ g(V,W)\eta(U)\big]\Big\}\xi + k\Big\{g(V,\phi W)\phi hU - g(U,\phi W)\phi hV + g(W,\phi hV)\phi U - \\ g(W,\phi hU)\phi V + \eta(W)\big[\eta(U)hV - \eta(V)hU\big]\Big\} - \mu\Big\{\eta(V)\big[(1-k)\eta(W)U + \\ \mu\eta(U)hW\big] - \eta(U)\big[(1-k)\eta(W)V + \mu\eta(V)hW\big] + 2g(U,\phi V)\phi hW\Big\}\Bigg] - \\ \frac{a}{2n-1}\big[S(V,hW)U - S(V,W)hU - S(U,hW)V + S(U,W)hV + \\ g(V,hW)QU - g(V,W)hQU - g(U,hW)QV + g(U,W)hQV\big] - \\ \frac{br}{2n}\big[g(V,hW)U - g(V,W)hU - g(U,hW)V + g(U,W)hV\big] = 0. \end{aligned} \tag{44}$$

Putting $U = hU$ in (44) and using (7), (13), we obtain

$$\begin{aligned} a\Big[-k(k-1)\eta(U)\eta(V)\eta(W)\xi + k(k-1)g(U,W)\eta(V)\xi + \\ \mu(k-1)g(hU,W)\eta(V)\xi - k(k-1)g(V,\phi W)\phi U - kg(hU,\phi W)\phi hV + \\ kg(W,\phi hV)\phi hU + k(k-1)g(W,\phi U)\phi V - k(k-1)\eta(U)\eta(V)\eta(W)\xi + \\ k(k-1)\eta(V)\eta(W)U - \mu(1-k)\eta(W)\eta(V)hU - 2\mu g(hU,\phi V)\phi hW\Big] - \\ \frac{a}{2n-1}\big[S(V,hW)hU - (k-1)\eta(U)S(V,W)\xi + (k-1)S(V,W)U - \\ 2nk(k-1)\eta(U)\eta(W)V + (k-1)S(U,W)V + S(hU,W)hV + g(V,hW)QhU - \\ g(V,W)hQhU - (k-1)\eta(U)\eta(W)QV + (k-1)g(U,W)QV + \\ g(hU,W)hQV\big] - \frac{br}{2n}\big[g(V,hW)hU - (k-1)g(V,W)\eta(U)\xi + \\ (k-1)g(V,W)U - (k-1)\eta(U)\eta(W)V + (k-1)g(U,W)V + \\ g(hU,W)hV\big] = 0. \end{aligned} \tag{45}$$

Taking an inner product with ξ in (45), we yield

$$\begin{aligned} S(U,W) = \frac{[2nak(2n+1) + br(2n-1)]}{2na}\eta(U)\eta(W) \\ -\frac{[2nak + br(2n-1)]}{2na}g(U,W) + \mu(2n-1)g(hU,W). \end{aligned} \tag{46}$$

Using (18) and (46), we obtain the following relation

$$S(U,W) = A_1 g(U,W) + B_1\eta(U)\eta(W), \tag{47}$$

where,

$$A_1 = \frac{(1-n)[2(2n-1)(\mu+br) - 4nak(\mu-1)] + \mu(2n-1)(2n\mu-k-br)}{4na(\mu-1)(1-n)},$$

and,

$$B_1 = \frac{(1-n)[4nak(k-1)(2n+1) - 2(2n-1)(2na\mu+br)]}{4na(\mu-1)(1-n)} + \frac{\mu(2n-1)[br(3-4n) - 2na(n\mu+k)]}{4na(\mu-1)(1-n)}.$$

Thus, M is η-Einstein manifold. □

In view of Lemma 1, we can state the following:

Corollary 2. *If $M^{2n+1}, (n>1)$ is a non-Sasakian h-semiconformally semi-symmetric (k,μ)-contact metric manifold with $\mu \neq 1$, then $Q\phi = \phi Q$ is satisfied.*

In view of (47), by simple substitution we can obtain the following theorem:

Theorem 5. *Let $M^{2n+1}, (n>1)$ be a non-Sasakian (k,μ)-contact metric manifold. The expression for the Ricci tensor of the manifold satisfying certain curvature conditions are as follows:*

Curvature condition	Expression for Ricci tensor
$H(U,V)\cdot h = 0$, (For $a=1, b=0$)	$S=\left[\frac{(1-n)[2\mu(2n-1)-4nak(\mu-1)]}{4n(\mu-1)(1-n)} + \frac{\mu(2n-1)(2n\mu-k)}{4n(\mu-1)(1-n)}\right]g + \left[\frac{(1-n)[4nk(k-1)(2n+1)-4n\mu(2n-1)]}{4n(\mu-1)(1-n)} - \frac{2n\mu(2n-1)(n\mu+k)}{4n(\mu-1)(1-n)}\right]\eta\otimes\eta$ η-Einstein manifold
$C(U,V)\cdot h = 0$, (For $a=1, b=-\frac{1}{2n-1}$)	$S=\left[\frac{(1-n)[2\mu(2n-1)-2r-4nk(\mu-1)]}{4n(\mu-1)(1-n)} + \frac{\mu[(2n\mu-k)(2n-1)+r]}{4n(\mu-1)(1-n)}\right]g + \left[\frac{(1-n)[4nk(k-1)(2n+1)-4n\mu(2n-1)+2r]}{4n(\mu-1)(1-n)} - \frac{\mu[r(3-4n)+2n(2n-1)(n\mu+k)]}{4n(\mu-1)(1-n)}\right]\eta\otimes\eta.$ η-Einstein manifold

6 ϕ-semiconformally semi-symmetric non-Sasakian (k,μ)-contact metric manifold

Definition 4. *A Riemannian manifold (M^{2n+1}, g) is said to be ϕ-semiconformally semi-symmetric if*

$$P(U,V)\cdot\phi = 0,$$

holds for any vector fields U, V on M.

Lemma 3. *[6] Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a contact manifold with ξ belonging to the (k,μ)-nullity distribution. Then, one has*

$$\begin{aligned}R(U,V)\phi W - \phi R(U,V)W = \big\{&(1-k)[g(\phi V,W)\eta(U) - g(\phi U,W)\eta(V)] + \\ &(1-\mu)[g(\phi hV,W)\eta(U) - g(\phi hU,W)\eta(V)]\big\}\xi - g(V+hV,W)(\phi U+\phi hU) + \\ &g(U+hU,W)(\phi V+\phi hV) - g(\phi V+\phi hV,W)(U+hU) + \\ &g(\phi U+\phi hU,W)(V+hV) - \eta(W)\big\{(1-k)[\eta(U)\phi V - \eta(V)\phi U] + \\ &(1-\mu)[\eta(U)\phi hV - \eta(V)\phi hU]\big\},\end{aligned} \quad (48)$$

for any vector fields U, V, W on M.

Theorem 6. *Let $M^{2n+1}, (n>1)$ be a non-Sasakian (k,μ)-contact metric manifold. If M is ϕ-semiconformally semi-symmetric with $\mu \neq \frac{n-1}{n}$, then M is η-Einstein manifold.*

Proof: Suppose M is ϕ-semiconformally semi-symmetric, i.e.,

$$(P(U,V)\cdot\phi)W = 0,$$

which implies,

$$P(U,V)\phi W - \phi P(U,V)W = 0. \tag{49}$$

Using (21) in (49), we get,

$$\begin{aligned} a\big[R(U,V)\phi W - \phi R(U,V)W\big] - \frac{a}{2n-1}\big[S(V,\phi W)U - S(V,W)\phi U - \\ S(U,\phi W)V + S(U,W)\phi V + g(V,\phi W)QU - g(V,W)\phi QU - \\ g(U,\phi W)QV + g(U,W)\phi QV\big] - \frac{br}{2n}\big[g(V,\phi W)U - \\ g(V,W)\phi U - g(U,\phi W)V + g(U,W)\phi V\big] = 0. \end{aligned} \tag{50}$$

Using (48) in (50), we obtain,

$$\begin{aligned} a\Big[\big\{(1-k)[g(\phi V,W)\eta(U) - g(\phi U,W)\eta(V)] + (1-\mu)[g(\phi hV,W)\eta(U) - \\ g(\phi hU,W)\eta(V)]\big\}\xi - g(V+hV,W)(\phi U + \phi hU) + \\ g(U+hU,W)(\phi V+\phi hV) - g(\phi V+\phi hV,W)(U+hU) + \\ g(\phi U+\phi hU,W)(V+hV) - \eta(W)\big\{(1-k)[\eta(U)\phi V - \eta(V)\phi U] + \\ (1-\mu)[\eta(U)\phi hV - \eta(V)\phi hU]\big\}\Big] - \frac{a}{2n-1}\big[S(V,\phi W)U - \\ S(V,W)\phi U - S(U,\phi W)V + S(U,W)\phi V + g(V,\phi W)QU - \\ g(V,W)\phi QU - g(U,\phi W)QV + g(U,W)\phi QV\big] - \frac{br}{2n}\big[g(V,\phi W)U - \\ g(V,W)\phi U - g(U,\phi W)V + g(U,W)\phi V\big] = 0. \end{aligned} \tag{51}$$

Putting $U = \phi U$ and making use of (7) in (51), we get,

$$\begin{aligned} a\Big[\big\{-(1-k)\eta(U)\eta V\eta W + (1-k)g(U,W)\eta(V) - (1-\mu)g(hU,W)\eta(V)\big\}\xi - \\ \eta(U)g(V+hV,W)\xi + g(V+hV,W)U - g(V+hV,W)hU + \\ g(\phi U+h\phi U,W)(\phi V+\phi hV) - g(\phi V+\phi hV,W)(\phi U+h\phi U) + \\ \eta(U)\eta(W)(V+hV) + g(hU,W)(V+hV) + (1-k)\eta(U)\eta(V)\eta(W)\xi - \\ (1-k)\eta(V)\eta(W)U - (1-\mu)\eta(V)hU\Big] - \frac{a}{2n-1}\big[S(V,\phi W)\phi U - \\ \eta(U)S(V,W)\xi + S(V,W)U - S(U,W)V + 2nk\eta(U)\eta(W)V + \\ 4(n-1)g(hU,W)V + S(\phi U,W)\phi V + g(V,\phi W)Q\phi U - g(V,W)\phi Q\phi U - \\ g(U,W)QV + \eta(U)\eta(W)QV + g(\phi U,W)\phi QV\big] - \frac{br}{2n}\big[g(V,\phi W)\phi U - \\ \eta(U)g(V,W)\xi + g(V,W)U - g(U,W)V + \eta(U)\eta(W)V + \\ g(\phi U,W)\phi V\big] = 0. \end{aligned} \tag{52}$$

Taking an inner product with ξ in (52) we get

$$\begin{aligned} S(U,W) = \frac{[2nak(2n+1) + br(2n-1)]}{2na}\eta(U)\eta(W) \\ -\frac{[2na(1-k)\{2n(1+k)-1\} + br(2n-1)]}{2na}g(U,W) \\ +[4(n-1) - \mu(2n-1)]g(hU,W). \end{aligned} \tag{53}$$

Using (18), (53) can be written as

$$S(U,W) = A_2 g(U,W) + B_2\eta(U)\eta(W), \tag{54}$$

where,

$$\begin{aligned} A_2 = \frac{[4(n-1)-\mu(2n-1)][2na\{k(2nk-1)+n(\mu-4)+3\} - br(2n-1)]}{4na(n\mu-n+1)} \\ -\frac{2(n\mu-n+1)[2na(1-k)(2nk+2n-1) + br(2n-1)]}{4na(n\mu-n+1)} \end{aligned}$$

and,

$$B_2 = \frac{[4(n-1) - \mu(2n-1)][(2n-1)(br - 2nak) - 4an(n\mu - n + 1)]}{4na(n\mu - n + 1)} + \frac{2(n\mu - n + 1)[2nak(2n+1) + br(2n-1)]}{4na(n\mu - n + 1)}.$$

Hence, this proves the theorem. □

In view of Lemma 1, we can state the following:

Corollary 3. *If $M^{2n+1}, (n > 1)$ is a non-Sasakian ϕ-semiconformally semi-symmetric (k, μ)-contact metric manifold with $\mu \neq \frac{n-1}{n}$, then $Q\phi = \phi Q$ is satisfied.*

In view of (54), one can obtain the following:

Theorem 7. *Let $M^{2n+1}, (n > 1)$ be a non-Sasakian (k, μ)-contact metric manifold. The expression for the Ricci tensor of the manifold satisfying certain curvature conditions are as follows:*

Curvature condition	Expression for Ricci tensor
$H(U,V) \cdot \phi = 0$, (For $a = 1, b = 0$)	$S = \left[\frac{[4(n-1) - \mu(2n-1)][2n\{k(2nk-1) + n(\mu-4) + 3\}]}{4n(n\mu - n + 1)} - \frac{2(n\mu - n + 1)[2n(1-k)(2nk + 2n - 1)]}{4n(n\mu - n + 1)}\right] g + \left[\frac{[49n-1) - \mu(2n-1)][-2nk(2n-1) - 4n(n\mu - n + 1)]}{4n(n\mu - n + 1)} + \frac{2(n\mu - n + 1)[2nk(2n+1)]}{4n(n\mu - n + 1)}\right] \eta \otimes \eta$ η-Einstein manifold
$C(U,V) \cdot \phi = 0)$ (For $a = 1, b = -\frac{1}{2n-1}$)	$S = \left[\frac{[4(n-1) - \mu(2n-1)][2n\{k(2nk-1) + n(\mu-4) + 3\} + r]}{4n(n\mu - n + 1} - \frac{2(n\mu - n + 1)[2n(1-k)(2nk + 2n - 1) - r]}{4n(n\mu - n + 1)}\right] g$ - $\left[\frac{[4(n-1) - \mu(2n-1)][(2nk(2n-1) + r) + 4n(n\mu - n + 1)]}{4n(n\mu - n + 1)} - \frac{2(n\mu - n + 1)[2nk(2n+1) - r]}{4n(n\mu - n + 1)}\right] \eta \otimes \eta$ η-Einstein manifold

7 A (k, μ)-contact metric manifold M^{2n+1} satisfying $R \cdot P = 0$

Definition 5. *A (k, μ)-contact metric manifold M^{2n+1} is said to be semiconformally semi-symmetric, if the semiconformal curvature tensor satisfies*

$$R(U,V) \cdot P = 0,$$

for all vector fields U, V in M.

Theorem 8. *Let $M^{2n+1}, (n > 1)$ be a non-Sasakian (k, μ)-contact metric manifold. If M is semiconformally semi-symmetric, then it is η-Einstein manifold, provided $\mu \neq 1$.*

Proof: Suppose M is semiconformally semi-symmetric,i.e.,

$$(R(\xi, U) \cdot P(V,W)X = 0.$$

which implies,

$$R(\xi, U)P(V,W)X - P(R(\xi, U)V, W)X \\ -P(V, R(\xi, U)W)X - P(V,W)R(\xi, U)X = 0. \tag{55}$$

Using (15) in (55), we get

$$k[g(U, P(V,W)X)\xi - \eta(P(V,W)X)U - g(U,V)P(\xi,W)X + \\ \eta(V)P(U,W)X - g(U,W)P(V,\xi)X + \eta(W)P(V,U)X - \\ g(U,X)P(V,W)\xi + \eta(X)P(V,W)U] + \mu[g(hU, P(V,W)X)\xi - \\ g(hU,V)P(\xi,W)X + \eta(V)P(hU,W)X + g(hU,W)P(\xi,V)X - \\ \eta(W)P(hU,V)X + \eta(X)P(V,W)hU - g(hU,X)P(V,W)\xi - \\ \eta(P(V,W)X)hU] = 0. \tag{56}$$

Putting $U = hU$ in (56), we have

$$k[g(hU, P(V,W)X)\xi - \eta(P(V,W)X)hU - g(hU,V)P(\xi,W)X + \eta(V)P(hU,W)X - g(hU,W)P(V,\xi)X + \eta(W)P(V,hU)X) - g(hU,X)P(V,W)\xi + \eta(X)P(V,W)hU] - \mu(k-1)[g(U,P(V,W)X)\xi - \eta(P(V,W)X)U - g(U,V)P(\xi,W)X + \eta(V)P(U,W)X - g(U,W)P(V,\xi)X + \eta(W)P(V,U)X - g(U,X)P(V,W)\xi + \eta(X)P(V,W)U] = 0. \quad (57)$$

From (56) and (57), we obtain

$$[k^2 + \mu^2(k-1)][g(U,P(V,W)X)\xi - \eta(P(V,W)X)U - g(U,V)P(\xi,W)X + \eta(V)P(U,W)X - g(U,W)P(V,\xi)X + \eta(W)P(V,U)X - g(U,X)P(V,W)\xi + \eta(X)P(V,W)U] = 0. \quad (58)$$

We know that for a non-Sasakian (k,μ)-contact metric manifold, $[k^2 + \mu^2(k-1)] \neq 0$. Taking an inner product with ξ in (58) we obtain

$$g(U,P(V,W)X) - \eta(P(V,W)X)\eta(U) - g(U,V)\eta(P(\xi,W)X) + \eta(V)\eta(P(U,W)X) - g(U,W)\eta(P(V,\xi)X) + \eta(W)\eta(P(V,U)X) - g(U,X)\eta(P(V,W)\xi) + \eta(X)\eta(P(V,W)U) = 0. \quad (59)$$

Contracting (60) with U over V, we obtain

$$\sum_{i=1}^{2n+1} \tilde{P}(e_i,W,X,e_i) - 2n\eta(P(\xi,W)X) + \sum_{i=1}^{2n+1} \eta(X)\eta(P(e_i,W)e_i) - \sum_{i=1}^{2n+1} \eta(W)\eta(P(e_i,e_i)X) - \eta(P(X,W)\xi) = 0. \quad (60)$$

Using (30), (60) reduces to

$$S(W,X) = \frac{[ar - 2nak]}{2na} g(W,X) - \frac{[ar - 2nak(2n+1)]}{2na}\eta(X)\eta(W) + \mu(2n-1)g(hW,X). \quad (61)$$

From (18) and (61), we obtain the following

$$\begin{aligned} S(W,X) &= \Bigg[\frac{[2(\mu-1)(r-2nk) + 4n\mu(2n-1)](n-1)}{4n(n-1)(\mu-1)} \\ &+ \frac{\mu(2n-1)[2(k-n\mu)-r]}{4n(n-1)(\mu-1)}\Bigg] g(W,X) \\ &+ \Bigg[\frac{[2r(1-\mu) - 4n(2n+1)](n-1) - 2n\mu k(2n+1)}{4n(n-1)(\mu-1)} \\ &+ \frac{\mu(2n-1)[2n9\mu + 4n - 2) + r]}{4n(n-1)(\mu-1)}\Bigg]\eta(W)\eta(X). \end{aligned} \quad (62)$$

Hence, the manifold is η-Einstein manifold, provided $\mu \neq 1$. □

In view of Lemma 1, we can state the following:

Corollary 4. *If $M^{2n+1}(n > 1)$, is a non-Sasakian semiconformally semi-symmetric (k,μ)-contact metric manifold with $\mu \neq 1$, then $Q\phi = \phi Q$ is satisfied.*

Acknowledgement

The authors are grateful to referees for their valuable suggestions toward the improvement of the paper. The second author is thankful to the Department of Science and Technology, New Delhi, India for financial support in the form of Inspire Fellowship (DST/INSPIRE Fellowship/2018/IF180830).

8 References

1 J. Kim, *A type of conformal curvature tensor*, Far East J. Math. Soc. **99**(1) (2016), 61-74.
2 J. Kim, *On pseudo semiconformally symmetric manifolds*, Bull. Korean Math. Soc. **54**(1) (2017), 177-186.
3 U. C. De, V. J. Suh, *On Weakly Semiconformally symmetric manifolds*, Acta Math. Hungar. **157**(2) (2019), 503-521.
4 J. P. Singh, M. Khatri, *On almost pseudo semiconformally symmetric manifolds*, Differ. Geom.-Dyn. Syst. **22** (2020), 233-253.
5 S. Tanno, *Ricci curvatures of contact Riemannian manifolds*, Tohoku Math. J. **40**(3) (1988), 441-448.
6 D. E. Blair, T. Koufogiorgos, B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. **19**(1-3) (1995), 189-214.
7 E. Boeckx, *A full classification of contact metric (k, μ)-spaces*, Illinois J. Math. **44** (2000), 212-219.
8 E. Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **54** (1926), 214-264.
9 O. Kowalski, *An explicit classification of 3-dimentional Riemannian spaces satisfying $R(U, V) \cdot R = 0$*, Czechoslovak Math. J. **46**(3) (1996), 427-474.
10 K. K. Baishya, P. R. Chowdhury, *On generalized quasi-conformal $N(k, \mu)$-manifolds*, Commun. Korean Math. Soc. **31**(1) (2018), 163-176.
11 A. A. Shaikh, K. Arslan, C. Murathan, K. K. Baishya, *On 3-dimensional generalized (k, μ)-contact manifolds*, Balkan J. Geom. Appl. **12**(1) (2007), 122-134.
12 K. K. Baishya, S. Eyasmin, A. A. Shaikh, *On ϕ-recurrent generalized (k, μ)-contact metric manifolds*, Lobachevskii J. Math. **27** (2007), 3-13.
13 U. C. De, J. Jun, S. Samui, *Certain semisymmetry properties of (k, μ)-contact metric manifolds*, Bull. Korean Math. Soc. **53**(4) (2016), 1237-1247.
14 D. G. Prakasha, K. K. Mirji, *On (k, μ)-Paracontact metric manifolds*, Gen. Math. Notes **25**(2) (2014), 68-77.
15 M. Khatri, J. P. Singh, *On a class of generalized recurrent (k, μ)-contact metric manifolds*, Commun. Korean Math. Soc. **35**(4) (2020), 1283-1297.
16 E. Boeckx, L. Vanhecke, *Characteristic reflection on unit tangent sphere bundle*, Houst. J. Math. **23** (1997), 427-448.
17 A. Sarkar, U. C. De, *On the quasi conformal curvature tensor of a (k, μ)-contact metric manifold*, Math. Rep. (Bucur.) **14**(2) (2012), 115-129.
18 D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture notes Math. (509) Springer, Berlin, 1976.
19 B. J. Papantoniou, *Contact Riemannian manifolds satisfying $R(\xi, U) \cdot R = 0$ and $\xi \in (k, \mu)$-nullity distribution*, Yokohama Math. J. **40**(2) (1993), 149-161.
20 A. Yildiz, U. C. De, *A classification of (k, μ)-contact metric manifolds*, Commun. Korean Math. Soc. **27**(2) (2012), 327-339.
21 S. Sasaki, *Almost-contact manifolds*, Part I Lecture Notes, Mathematical Institue Tohoku Univ., 1965.

Conference Proceeding Science and Technology, 4(2), 2021, 226–231

18th International Geometry Symposium, 2021.

On the Geometry of φ-Fixed Points

ISSN: 2651-544X
http://cpostjournal.org/

Nihal Özgür[1,*] *Nihal Taş*[2]

[1] *Department of Mathematics, Faculty of Science and Arts, Balıkesir University, Balıkesir, Turkey, ORCID:0000-0002-8152-1830*
[2] *Department of Mathematics, Faculty of Science and Arts, Balıkesir University, Balıkesir, Turkey, ORCID:0000-0002-4535-4019*
** Corresponding Author E-mail: nihal@balikesir.edu.tr*

Abstract: In this study, we present some solutions to an open problem related to the geometric study of φ-fixed points arisen in a recent paper. Our methods depend on the usage of appropriate auxiliary numbers such as $M(u,v)$ defined by

$$M(u,v)=\max\left\{d(u,v),d(u,Tu),d(v,Tv),\left[\frac{d(u,Tv)+d(v,Tu)}{1+d(u,Tu)+d(v,Tv)}\right]d(u,v)\right\},$$

for all $u,v\in X$, and

$$\rho=\inf\left\{d\left(Tu,u\right):u\in X,Tu\neq u\right\}.$$

Keywords: Fixed point, φ-fixed point, φ-fixed circle, φ-fixed disc.

1 Introduction and Motivation

The notion of a φ-fixed point was introduced in [1]. An element $u\in X$ is said to be a φ-fixed point of the self-mapping $T:X\to X$, where $\varphi:X\to[0,\infty)$ is a given function, if u is a fixed point of T and $\varphi\left(u\right)=0$ [1]. Several existence results of φ-fixed points for various classes of operators have been studied (see for example [1–9]). Following [1], we denote the set of all zeros of the function φ by Z_{φ}, that is, we have

$$Z_{\varphi}=\left\{u\in X:\varphi\left(u\right)=0\right\}.$$

Recently, an open problem related to the geometric properties of non-unique φ-fixed points was stated in [10]. Let $T:X\to X$ be a self-mapping on a metric space (X,d) and $Fix(T)=\{u\in X:Tu=u\}$ be the fixed point set of T. In [10], in the context of the fixed-circle problem (resp. fixed-disc problem), it was pointed out that new results on the geometric properties of the φ-fixed points of a self-mapping can be investigated via the help of appropriate auxiliary numbers. The new notions of a φ-fixed circle and of a φ-fixed disc were defined as follows:

Definition 1. *[10] Let (X,d) be a metric space, T be a self-mapping of X and $\varphi:X\to[0,\infty)$ be a given function.*
1) A circle $C_{u_0,r}=\{u\in X:d\left(u,u_0\right)=r\}$ in X is said to be a φ-fixed circle of T if and only if $C_{u_0,r}\subseteq Fix(T)\cap Z_{\varphi}$.
2) A disc $D_{u_0,r}=\{u\in X:d\left(u,u_0\right)\leq r\}$ in X is said to be a φ-fixed disc of T if and only if $D_{u_0,r}\subseteq Fix(T)\cap Z_{\varphi}$.

For examples and more details we refer the reader to [10]. The proposed open problem is the investigation of the existence and uniqueness of φ-fixed circles (resp. φ-fixed discs) for various classes of self-mappings. In [11], some solutions to this problem were presented via the help of appropriate auxiliary numbers and geometric conditions. It was proved that any zero of a given function $\varphi:X\to[0,\infty)$ can produce a fixed circle (resp. fixed disc) contained in the set $Fix\left(T\right)\cap Z_{\varphi}$ for a self-mapping T on a metric space. To do this, new types of a contraction such as a type 1 φ_{u_0}-contraction (resp. type 2 φ_{u_0}-contraction, type 3 φ_{u_0}-contraction) and a generalized type 1 φ_{u_0}-contraction (resp. generalized type 2 φ_{u_0}-contraction, generalized type 3 φ_{u_0}-contraction) were defined.

In this paper, we define new types of φ_{u_0}-contractions to present new solutions to the φ-fixed circle problem (resp. φ-fixed disc problem). Our main tools are the numbers

$$M(u,v)=\max\left\{d(u,v),d(u,Tu),d(v,Tv),\left[\frac{d(u,Tv)+d(v,Tu)}{1+d(u,Tu)+d(v,Tv)}\right]d(u,v)\right\},\tag{1}$$

defined for all $u,v\in X$ and

$$\rho:=\inf\left\{d\left(Tu,u\right):u\in X,u\neq Tu\right\},\tag{2}$$

$$\mu:=\inf\left\{\sqrt{d\left(Tu,u\right)}:u\in X,u\neq Tu\right\}.\tag{3}$$

2 φ-Fixed Circle and φ-Fixed Disc Results

In this section, we give new φ-fixed circle (resp. φ-fixed disc) results using the auxiliary numbers $M(u,v)$, ρ and μ defined in (1), (2) and (3), respectively, together with geometric conditions.

First, we define a new type of a φ_{u_0}-contraction on a metric space.

Definition 2. *Let (X,d) be a metric space, T be a self-mapping of X and $\varphi : X \to [0,\infty)$ be a given function. If there exists a point $u_0 \in X$ such that*

$$d(Tu,u) > 0 \Rightarrow \max\{d(u,Tu), \varphi(Tu)\} + \ln(\varphi(u)+1) \leq k \max\{d(u,u_0), \varphi(u)\} + \ln(\varphi(u_0)+1), \tag{4}$$

for all $u \in X$ and some $k \in (0,1)$, then T is called a type 4 φ_{u_0}-contraction.

In the following theorem, we see that the number ρ defined in (2) and the point u_0 produce a φ-fixed circle (resp. φ-fixed disc) of a type 4 φ_{u_0}-contraction T under a geometric condition.

Theorem 1. *Let (X,d) be a metric space, the number ρ be defined as in (2) and $T : X \to X$ be a type 4 φ_{u_0}-contraction with the point $u_0 \in X$ and the given function $\varphi : X \to [0,\infty)$. If $u_0 \in Z_\varphi$ and*

$$\varphi(u) \leq d(Tu,u), \tag{5}$$

for all $u \in C_{u_0,\rho}$, then the circle $C_{u_0,\rho}$ is a φ-fixed circle of T.

Proof: To show $u_0 \in Fix(T)$, conversely, suppose that $u_0 \neq Tu_0$. Then we have $d(Tu_0,u_0) > 0$ and using the inequality (4) together with the hypothesis $u_0 \in Z_\varphi$, we find

$$\max\{d(u_0,Tu_0), \varphi(Tu_0)\} + \ln(\varphi(u_0)+1) \leq k \max\{d(u_0,u_0), \varphi(u_0)\} + \ln(\varphi(u_0)+1) = 0,$$

and hence

$$\max\{d(u_0,Tu_0), \varphi(Tu_0)\} = 0.$$

This implies $d(u_0,Tu_0) = 0$, which is a contradiction with our assumption. Hence we obtain $Tu_0 = u_0$ and $u_0 \in Fix(T) \cap Z_\varphi$.

If $\rho = 0$, then clearly $C_{u_0,\rho} = \{u_0\} \subset Fix(T) \cap Z_\varphi$ and hence, the circle $C_{u_0,\rho}$ is a φ-fixed circle of T.

Let $\rho > 0$. For any $u \in C_{u_0,\rho}$ with $Tu \neq u$, we have

$$\begin{aligned} \max\{d(u,Tu), \varphi(Tu)\} + \ln(\varphi(u)+1) &\leq k \max\{d(u,u_0), \varphi(u)\} + \ln(\varphi(u_0)+1) \\ &= k \max\{\rho, \varphi(u)\}. \end{aligned}$$

If $\max\{\rho, \varphi(u)\} = \rho$, then by the definition of the number ρ, we get

$$\max\{d(u,Tu), \varphi(Tu)\} + \ln(\varphi(u)+1) \leq k\rho \leq kd(u,Tu)$$

and so $d(u,Tu) \leq kd(u,Tu)$, a contradiction by the hypothesis $k \in (0,1)$.

If $\max\{\rho, \varphi(u)\} = \varphi(u)$, we get

$$\max\{d(u,Tu), \varphi(Tu)\} + \ln(\varphi(u)+1) \leq k\varphi(u)$$

and hence $d(u,Tu) \leq k\varphi(u)$, a contradiction because of (5) and the hypothesis $k \in (0,1)$.

This contradiction shows that $Tu = u$, that is, $u \in Fix(T)$. By (5), we have $\varphi(u) = 0$ for all $u \in C_{u_0,\rho}$. This means $u \in Fix(T) \cap Z_\varphi$ for all $u \in C_{u_0,\rho}$. Consequently, we find $C_{u_0,\rho} \subset Fix(T) \cap Z_\varphi$ and hence, the circle $C_{u_0,\rho}$ is a φ-fixed circle of T. □

Theorem 2. *Let (X,d) be a metric space, the number ρ be defined as in (2), $T : X \to X$ be a type 4 φ_{u_0}-contraction with the point $u_0 \in X$ and the given function $\varphi : X \to [0,\infty)$. If $u_0 \in Z_\varphi$ and the inequality*

$$\varphi(u) \leq d(Tu,u)$$

is satisfied for all $u \in D_{u_0,\rho}$, then the disc $D_{u_0,\rho}$ is a φ-fixed disc of T.

Proof: The proof is similar to the proof of Theorem 1, so we omit it. □

We give an illustrative example for Theorem 1 and Theorem 2.

Example 1. *Let $(\mathbb{R}, d)$ be the usual metric space and consider the self-mapping $T : X \to X$ defined by*

$$Tu = \begin{cases} u + \frac{1}{2} & ; \quad u > 2 \\ u & ; \quad u \leq 2 \end{cases}$$

and the function $\varphi : \mathbb{R} \to [0, \infty)$ defined by

$$\varphi(u) = \begin{cases} 0 & ; \quad u \geq -1 \\ |u| & ; \quad u < -1 \end{cases}.$$

We show that T is a type 4 φ_{u_0}-contraction with the point $u_0 = 0$ and $k = \frac{2}{3}$. Indeed, we have

$$\max\left\{\left|u - \left(u + \frac{1}{2}\right)\right|, \varphi\left(u + \frac{1}{2}\right)\right\} + \ln(0+1) = \frac{1}{2} \leq \frac{2}{3}\max\{|u-0|, \varphi(u)\} + \ln(0+1) = \frac{2}{3}u,$$

for all $u > 2$. We find

$$\begin{aligned} \rho &= \inf\{d(Tu, u) : u \in X, u \neq Tu\} \\ &= \inf\left\{\left|u + \frac{1}{2} - u\right| = \frac{1}{2} : u > 2\right\} \\ &= \frac{1}{2} \end{aligned}$$

and all conditions of Theorem 1 are satisfied by T. Observe that $Fix(T) \cap Z_\varphi = (-\infty, 2] \cap [-1, \infty) = [-1, 2]$ and we get

$$C_{0,\frac{1}{2}} = \left\{-\frac{1}{2}, \frac{1}{2}\right\} \subset Fix(T) \cap Z_\varphi.$$

Hence, the circle $C_{0,\frac{1}{2}}$ is a φ-fixed circle of T.

Clearly, T also satisfies all conditions of Theorem 2 and we have $D_{0,\frac{1}{2}} \subset Fix(T) \cap Z_\varphi$. That is, the disc $D_{0,\frac{1}{2}} = \left[-\frac{1}{2}, \frac{1}{2}\right]$ is a φ-fixed disc of T.

Now, we define another type of a φ_{u_0}-contraction using the auxiliary number $M(u, v)$.

Definition 3. *Let (X, d) be a metric space, T be a self-mapping of X and $\varphi : X \to [0, \infty)$ be a given function. If there exists a point $u_0 \in X$ such that*

$$d(Tu, u) > 0 \Rightarrow \max\{d(u, Tu), \varphi(Tu)\} + \ln(\varphi(u) + 1) \leq k\max\{M(u, u_0), \varphi(u)\} + \ln(\varphi(u_0) + 1), \tag{6}$$

for all $u \in X$ and some $k \in \left(0, \frac{1}{2}\right)$, then T is called a generalized type 4 φ_{u_0}-contraction.

We prove the following φ-fixed circle theorem by means of the number μ. We note that we have

$$\begin{aligned} M(u_0, u_0) &= \max\left\{d(u_0, u_0), d(u_0, Tu_0), d(u_0, Tu_0), \left[\frac{d(u_0, Tu_0) + d(u_0, Tu_0)}{1 + d(u_0, Tu_0) + d(u_0, Tu_0)}\right] d(u_0, u_0)\right\} \\ &= d(u_0, Tu_0). \end{aligned}$$

Theorem 3. *Let (X, d) be a metric space, the number μ be defined as in (3) and $T : X \to X$ be a generalized type 4 φ_{u_0}-contraction with the point $u_0 \in X$ and the given function $\varphi : X \to [0, \infty)$. If $u_0 \in Z_\varphi$ and the inequalities*

$$d(Tu, u_0) \leq \mu, \tag{7}$$

$$\varphi(u) \leq d(Tu, u)$$

hold for all $u \in C_{u_0,\mu}$, then the circle $C_{u_0,\mu}$ is a φ-fixed circle of T.

Proof: If $d(Tu_0, u_0) > 0$, then using the inequality (6) and the hypothesis $u_0 \in Z_\varphi$, we find

$$\begin{aligned} \max\{d(u_0, Tu_0), \varphi(Tu_0)\} + \ln(\varphi(u_0) + 1) &\leq k\max\{M(u_0, u_0), \varphi(u_0)\} + \ln(\varphi(u_0) + 1) \\ &\leq kM(u_0, u_0) \\ &= kd(u_0, Tu_0) \end{aligned}$$

and hence

$$\max\{d(u_0, Tu_0), \varphi(Tu_0)\} \leq kd(u_0, Tu_0).$$

This implies $d(u_0, Tu_0) \leq kd(u_0, Tu_0)$, which is a contradiction since $k \in \left(0, \frac{1}{2}\right)$. Then, we have $Tu_0 = u_0$, that is, $u_0 \in Fix(T)$. Therefore, we obtain $u_0 \in Fix(T) \cap Z_\varphi$.

If $\mu = 0$, then clearly we find $C_{u_0,\mu} = \{u_0\} \subset Fix(T) \cap Z_\varphi$ and hence, the circle $C_{u_0,\mu}$ is a φ-fixed circle of T.

Let $\mu > 0$. For any $u \in C_{u_0,\mu}$ with $Tu \neq u$, we have

$$\max\{d(u,Tu), \varphi(Tu)\} + \ln(\varphi(u)+1) \leq k \max\{M(u,u_0), \varphi(u)\}. \tag{8}$$

If $\max\{M(u,u_0), \varphi(u)\} = \varphi(u)$, we obtain

$$\max\{d(u,Tu), \varphi(Tu)\} + \ln(\varphi(u)+1) \leq k\varphi(u)$$

and hence $d(u,Tu) \leq k\varphi(u)$ which is a contradiction by the hypothesis $k \in \left(0, \frac{1}{2}\right)$ and $\varphi(u) \leq d(Tu,u)$.

Let $\max\{M(u,u_0), \varphi(u)\} = M(u,u_0)$. Considering the condition (7), we obtain

$$\begin{aligned}
M(u,u_0) &= \max\left\{d(u,u_0), d(u,Tu), d(u_0,Tu_0), \left[\frac{d(u,Tu_0)+d(u_0,Tu)}{1+d(u,Tu)+d(u_0,Tu_0)}\right] d(u,u_0)\right\} \\
&\leq \max\left\{\mu, d(u,Tu), 0, \left[\frac{\mu+\mu}{1+d(u,Tu)}\right]\mu\right\} \\
&= \max\left\{\mu, d(u,Tu), 0, \frac{2\mu^2}{1+d(u,Tu)}\right\}.
\end{aligned}$$

If $\max\left\{\mu, d(u,Tu), 0, \frac{2\mu^2}{1+d(u,Tu)}\right\} = d(u,Tu)$ then by (8) we get

$$\max\{d(u,Tu), \varphi(Tu)\} + \ln(\varphi(u)+1) \leq kd(u,Tu)$$

and so $d(u,Tu) \leq kd(u,Tu)$, a contradiction by the hypothesis $k \in \left(0, \frac{1}{2}\right)$. If $M(u,u_0) \leq \frac{2\mu^2}{1+d(u,Tu)}$ then by the definition of the number μ we have

$$\begin{aligned}
\max\{d(u,Tu), \varphi(Tu)\} + \ln(\varphi(u)+1) &\leq k\frac{2\mu^2}{1+d(u,Tu)} \\
&\leq \frac{2k\left(\sqrt{d(u,Tu)}\right)^2}{1+d(u,Tu)} \\
&< 2kd(u,Tu)
\end{aligned}$$

and hence $d(u,Tu) < 2k(u,Tu)$, a contradiction by the hypothesis $k \in \left(0, \frac{1}{2}\right)$. This implies $Tu = u$, that is, $u \in Fix(T)$ in all of the above cases. By the hypothesis $\varphi(u) \leq d(Tu,u)$, we have $\varphi(u) = 0$ for all $u \in C_{u_0,\mu}$. This implies $u \in Fix(T) \cap Z_\varphi$ for all $u \in C_{u_0,\mu}$. Consequently, we deduce that $C_{u_0,\mu} \subset Fix(T) \cap Z_\varphi$ and hence, the circle $C_{u_0,\mu}$ is a φ-fixed circle of T. □

Theorem 4. *Let (X,d) be a metric space, the number μ be defined as in (3) and $T : X \to X$ be a type 4 generalized φ_{u_0}-contraction with the point $u_0 \in X$ and the given function $\varphi : X \to [0,\infty)$. If $u_0 \in Z_\varphi$ and the inequalities*

$$d(Tu,u_0) \leq \mu,$$

$$\varphi(u) \leq d(Tu,u)$$

hold for all $u \in D_{u_0,\mu}$, then the disc $D_{u_0,\mu}$ is a φ-fixed disc of T.

Proof: The proof is similar to the proof of Theorem 3. □

Now, in the following example, we see a sample of a self-mapping which is not a type 4 φ_{u_0}-contraction, but is a generalized type 4 φ_{u_0}-contraction.

Example 2. *Let $(\mathbb{R},d)$ be the usual metric space and consider the self-mapping $T : X \to X$ defined by*

$$Tu = \begin{cases} 2u & ; \quad u > 4 \\ u & ; \quad u \leq 4 \end{cases}$$

and the function $\varphi : \mathbb{R} \to [0,\infty)$ defined by

$$\varphi(u) = \begin{cases} 0 & ; \quad u \geq -5 \\ |u| & ; \quad u < -5 \end{cases}.$$

We have

$$\begin{aligned}
\rho &= \inf\{d(Tu,u) : u \in X, u \neq Tu\} \\
&= \inf\{|2u-u| : u > 4\} \\
&= \inf\{|u| : u > 4\} = 4
\end{aligned}$$

and

$$\begin{aligned}\mu &= \inf\left\{\sqrt{d(Tu,u)} : u\in X, u\neq Tu\right\}\\ &= \inf\left\{\sqrt{|2u-u|} : u>4\right\}\\ &= \inf\left\{\sqrt{|u|} : u>4\right\} = 2.\end{aligned}$$

We show that T is not a type 4 φ_{u_0}-contraction with the point $u_0 = 0$ and any $k \in (0,1)$. Indeed, we have

$$\max\{d(u,Tu), \varphi(Tu)\} + \ln(\varphi(u)+1) = \max\{|u|, 0\} + \ln(0+1) = |u|$$

and

$$\max\{d(u,0), \varphi(u)\} + \ln(\varphi(0)+1) = \max\{|u-0|, 0\} = |u|,$$

for all $u > 4$. These implies $|u| \leq k|u|$ which is a contradiction since $k \in (0,1)$. Therefore, the condition (4) is not satisfied by T.

On the other hand, we have

$$M(u,0) = \max\left\{|u|, |u|, 0, \left[\frac{|u|+|2u|}{1+|u|}\right]|u|\right\} = \frac{3|u|}{1+|u|}|u|,$$

for all $u > 4$. If we choose $k = \frac{5}{12}$, then (6) is satisfied and so T is a generalized type 4 φ_{u_0}-contraction with the point $u_0 = 0$.

Observe that

$$Fix(T)\cap Z_\varphi = (-\infty,4]\cap[-5,\infty)\cap = [-5,4]$$

and we get

$$C_{0,2} = \{-2,2\} \subset Fix(T)\cap Z_\varphi.$$

Hence, the circle $C_{0,2}$ is a φ-fixed circle of T.

Clearly, T also satisfies all conditions of Theorem 4 and the disc $D_{0,2} = [-2,2]$ is a φ-fixed disc of T.

Now, we give an example of a self-mapping T such that T satisfies all conditions of Theorem 1, Theorem 2, Theorem 3 and Theorem 4.

Example 3. *Let $X = \{-1,0,1,2,3\}$ with the usual metric and consider the self-mapping $T : X \to X$ defined by*

$$Tu = \begin{cases} u & , \quad u \neq 3 \\ 2 & ; \quad u = 3 \end{cases}$$

and the function $\varphi : X \to [0,\infty)$ defined by

$$\varphi(u) = u^3 - u.$$

We have

$$\begin{aligned}\rho &= \inf\{d(Tu,u) : u\in X, u\neq Tu\}\\ &= |2-3| = 1\end{aligned}$$

and

$$\begin{aligned}\mu &= \inf\left\{\sqrt{d(Tu,u)} : u\in X, u\neq Tu\right\}\\ &= \sqrt{1} = 1.\end{aligned}$$

Observe that $Fix(T) = X - \{3\}$, $Z_\varphi = \{-1,0,1\}$ and $Fix(T)\cap Z_\varphi = \{-1,0,1\}$.

Now, we show that T is a type 4 φ_{u_0}-contraction with the point $u_0 = 0$ and $k = \frac{1}{2}$. Indeed, we have

$$\max\{1,6\} + \ln(25) = 6 + \ln(25) \leq \frac{1}{2}\max\{|3-0|, 24\} = 12,$$

for $u = 3$. Clearly, all conditions of Theorem 1 (resp. Theorem 2) are satisfied by T. We get

$$C_{0,1} = \{-1,1\} \subset Fix(T)\cap Z_\varphi \text{ and } D_{0,1} = \{-1,0,1\} \subset Fix(T)\cap Z_\varphi.$$

Hence, the circle $C_{0,1}$ (resp. the disc $D_{0,1}$) is a φ-fixed circle (resp. φ-fixed disc $D_{0,1}$) of T.

On the other hand, T is a generalized type 4 φ_{u_0}-contraction with the point $u_0 = 0$ and $k = \frac{2}{5}$. Indeed, we have

$$\begin{aligned}M(3,0) &= \max\left\{d(3,0), d(3,2), d(0,0), \left[\frac{d(3,0)+d(0,2)}{1+d(3,2)+d(0,-0)}\right]d(3,0)\right\}\\ &= \max\left\{3,1,0,\frac{3+2}{2}.3\right\} = \frac{15}{2}\end{aligned}$$

and so,

$$\max\{1,6\} + \ln(25) = 6 + \ln(25) \leq \frac{2}{5}\max\left\{\frac{15}{2}, 24\right\} = \frac{48}{5}.$$

Clearly, T also satisfies all conditions of Theorem 3 (resp. Theorem 4).

Remark 1. 1) *Example* 2 *shows that the converse statements of Theorem* 1 *and Theorem* 2 *are not true everywhen. Notice that the circle* $C_{0,4}$ $($*resp. the disc* $D_{0,4})$ *is a* φ*-fixed circle* $($*resp. a* φ*-fixed disc*$)$ *of* T*. But, we have seen that* T *is not a type* 4 φ_{u_0}*-contraction with the point* $u_0 = 0$ *and any* $k \in (0, 1)$.

2) *Theorem* 1 *and Theorem* 3 $($*resp. Theorem* 2 *and Theorem* 4$)$ *can generate the same* φ*-fixed circle* $($*resp.* φ*-fixed disc*$)$ *of* T $($*see Example* 3$)$.

3 Conclusion and Future Work

We have given new solutions to the φ-fixed circle (resp. φ-fixed disc) problem with necessary illustrative examples. To do this, we were inspired by the definition of the function

$$F : [0, \infty)^3 \to [0, \infty), F(a, b, c) = \max\{a, b\} + \ln(c + 1).$$

As a future work, using the other examples of functions F on page 1 in [6], new φ-fixed circle (resp. φ-fixed disc) results can be obtained by similar techniques.

Acknowledgement

This work is supported by the Scientific Research Projects Unit of Balıkesir University under the project number 2020/019.

4 References

1 M. Jleli, B. Samet, C. Vetro, *Fixed point theory in partial metric spaces via φ-fixed point's concept in metric spaces*, J. Inequal. Appl., **2014** (2014), Article Number 426, 9 pages. https://doi.org/10.1186/1029-242X-2014-426.
2 P. Agarwal, M. Jleli, B. Samet, *On fixed points that belong to the zero set of a certain function*, Fixed Point Theory in Metric Spaces, Springer, Singapore, 2018. https://doi.org/10.1007/978-981-13-2913-5_7.
3 M. U. Ali, K. Muangchoo-in, P. Kumam, *Zeroes and Fixed Points of Different Functions via Contraction Type Conditions*, In: Anh L., Dong L., Kreinovich V., Thach N. (eds) Econometrics for Financial Applications. ECONVN 2018. Studies in Computational Intelligence, vol 760. Springer, Cham. https://doi.org/10.1007/978-3-319-73150-6_28.
4 D. Gopal, L. M. Budhia, S. Jain, *A relation theoretic approach for φ-fixed point result in metric space with an application to an integral equation*, Communications in Nonlinear Analysis, **6**(1) (2019), 89-95.
5 M. Imdad, A. R. Khan, H. N. Saleh, W. M. Alfaqih, *Some φ-fixed point results for $(F, \varphi, \alpha - \psi)$-contractive type mappings with applications*, Mathematics, **7**(2) (2019), 122.
6 E. Karapinar, D. O'Regan, B. Samet, *On the existence of fixed points that belong to the zero set of a certain function*, Fixed Point Theory Appl., **2015** (2015), Article Number 152, 14 pages.
7 P. Kumrod, W. Sintunavarat, *A new contractive condition approach to φ-fixed point results in metric spaces and its applications*, J. Comput. Appl. Math., **311** (2017), 194-204.
8 P. Kumrod, W. Sintunavarat, *On new fixed point results in various distance spaces via φ-fixed point theorems in D-generalized metric spaces with numerical results*, J. Fixed Point Theory Appl., **21**(3) (2019), Paper No. 86, 14 pages.
9 F. Vetro, *Fixed points that are zeros of a given function*, In: Cho Y.J., Jleli M., Mursaleen M., Samet B., Vetro C. (eds) Advances in Metric Fixed Point Theory and Applications. Springer, Singapore, 2021. https://doi.org/10.1007/978-981-33-6647-3_8.
10 N. Özgür, N. Taş, *New discontinuity results at fixed point on metric spaces*, J. Fixed Point Theory Appl., **23**(2) (2021), Paper No. 28, 14 pages.
11 N. Özgür, N. Taş, *φ-fixed points of self-mappings on metric spaces with a geometric viewpoint*, (2021), arXiv:2107.11199 [math.GN].

Conference Proceeding Science and Technology, 4(2), 2021, 232–237

18th International Geometry Symposium, 2021.

Images of Some Discs Under the Linear Fractional Transformation of Special Continued Fractions

ISSN: 2651-544X
http://cpostjournal.org/

Ümmügülsün Akbaba[1,*] *Tuğba Tuylu*[2] *Ali Hikmet Değer*[3]

[1] *Department of Mathematics, Faculty of Science , Karadeniz Technical University, Trabzon, Turkey, ORCID:0000-0002-5870-6802*
[2] *Department of Mathematics, Faculty of Science, Karadeniz Technical University, Trabzon, Turkey, ORCID:0000-0003-0354-0646*
[3] *Department of Mathematics, Faculty of Science, Karadeniz Technical University, Trabzon, Turkey, ORCID:0000-0003-0764-715X*
** Corresponding Author E-mail: ummugulsun.akbaba@ktu.edu.tr*

Abstract: In this paper, considering that the continued fraction $\mathbf{K}_n(-1/-k)$ is a Pringsheim fraction, with $n = 1, 2, 3, ...$ for natural numbers k that are $k = 2$ and $k = 3$. The forms of the images of the discs $\mathbf{K}_n$ are examined under the linear fractional transformations $\{S_n\}$ of the complex disc $\overline{\mathbb{D}} = \{w \in \mathbb{C} : |w| \leq 1\}$. Specially, the relation between Fibonacci numbers and forms of the images of the $\mathbf{K}_n$ are examined for $k = 3$. The results for these special continued fractions from the images of these discs will also be compared with the vertex values on the minimal-length paths in the suborbital graphs. Also, an algorithm is created in Python application language to visually inspect circular discs.

Keywords: Continued fractions, Fibonacci numbers, Pringsheim.

1 Introduction

Let $\varphi_i = \begin{pmatrix} -u & \frac{u^2+(-1)^i k_i u+1}{N} \\ -N & u+(-1)^i k_i \end{pmatrix} \in \Gamma_0(N), i = 1, 2$. If $(u, N) = 1$, then there exist an integer $k_i,\ i = 1, 2$ such that $u^2 + (-1)^i k_i u + 1 \equiv 0\ (modN), i = 1, 2$. On $\mathbf{F}_{u,N}$, φ_i, $i = 1, 2$ is a transformation which joins the vertices to each other by respectively on the infinite path of minimal length to both directions. We can see the vertices of the minimal length path of suborbital graph $\mathbf{F}_{u,N}$ at Figure 1 [1–3]. Here,

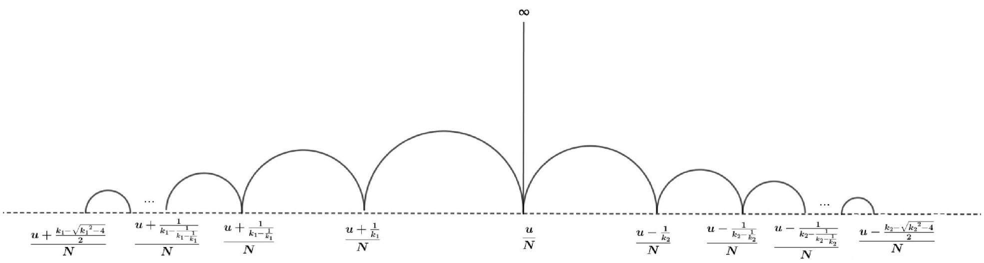

Fig. 1: The vertices on the minimal length path on the suborbital graph $\mathbf{F}_{u,N}$

every vertices on the minimal-length paths of suborbital graph $\mathbf{F}_{u,N}$ include a continued fraction.It is well known that a continued fraction may be regarded as a sequence of Möbius maps. We saw that the set M of vertices were obtained by a sequence of Möbius maps. So, there is naturally a connection between them. We know any continued fraction can be expressed as the symbol $b_0 + \mathrm{K}_{m=1}^{\infty}(a_m/b_m)$ by [4]. Using the terminology in [4], the n^{th} numerator A_n and the n^{th} denominator B_n of a continued fraction $b_0 + \mathrm{K}(a_m/b_m)$ are defined by the recurrence relations (second order linear difference equations)

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} := b_n \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} + a_n \begin{bmatrix} A_{n-2} \\ B_{n-2} \end{bmatrix}, \tag{1}$$

where $n = 1, 2, 3, \ldots$ with initial conditions $A_{-1} := 1, B_{-1} := 0, A_0 := b_0, B_0 := 1$. The modified approximant $S_n(z_n)$ can then be written as $S_n(z_n) = \frac{A_n + A_{n-1}z_n}{B_n + B_{n-1}z_n}$, where $n = 0, 1, 2, 3, \ldots$ and hence for the n^{th} approximant f_n we have $f_n = S_n(0) = \frac{A_n}{B_n}, f_{n-1} = S_n(\infty) = \frac{A_{n-1}}{B_{n-1}}$.

The continued fraction related the suborbital graphs is $\mathrm{K}_{n=1}^{\infty}\left(\frac{-1}{-k_i}\right)$, where $a_n = -1$, $b_n = -k_i$ for all $n \geq 0$, $n \in \mathbb{N}$ and $i = 1, 2$. That continued fraction is a Pringsheim continued fraction according to Pringsheim theorem [5]. From recurrence relation (1), for this continued fraction, $B_n = -A_{n+1}$. Then, n^{th} vertex on the path of minimal length in the suborbital graph $\mathbf{F}_{u,N}$ can be given by

$$\frac{u + (-1)^i T_n(0)}{N} = \frac{u + (-1)^i \frac{A_n}{B_n}}{N} = \frac{A_{n+1}u - (-1)^i A_n}{A_{n+1}N},\ i = 1, 2 \tag{2}$$

for left and right direction respectively.

Corollary 1. *[2] If $k_i \geq 2$, $i = 1, 2$, then from the linear equations* (1)*, we have recurrence relation as $k_i A_{n+1} + A_{n+2} + A_n = 0$.* ■

Theorem 1. *[2] If $k_i = 2$, $i = 1, 2$, then $A_n = (-1)^n n$ and if $k_i > 2$, then*

$$A_n = (-1)^n 2^{1-n} \sum_{t=1}^{n} \left(k_i + \sqrt{k_i^2 - 4}\right)^{n-t} \left(k_i - \sqrt{k_i^2 - 4}\right)^{t-1}. \tag{3}$$

Corollary 2. *[2] If $k_i = 3$, $i = 1, 2$, then $A_n = (-1)^n F_{2n}$, where F_n is the n^{th} Fibonacci number.*

Corollary 3. *If $k_i = 3$, $i = 1, 2$, then the $(n+1)^{th}$ vertex on the path of the minimal length starting with the vertex $\frac{u}{N}$ is $\frac{F_{2n} - F_{2n+2}u}{F_{2n+2}N}$ and $\frac{F_{2n} + F_{2n+2}u}{F_{2n+2}N}$, respectively, where, for each $m \in \mathbb{N} \cup \{0\}$,*

$$F_m = \begin{cases} 0, & if\ m=0; \\ 1, & if\ m=1; \\ F_{m-1} + F_{m-2}, & if\ m>1 \end{cases}$$

is the m^{th} Fibonacci number.

2 Main Results

The continued fractions are defined by the value regions V_n and the elements Ω_n corresponding to these regions. So, for all $\langle a_n, b_n \rangle \in \Omega_n$

$$s_n(v_n) \subseteq V_{n-1}, \quad n = 1, 2, 3, \dots,$$

where $\{\Omega\}$ is the sequence of element regions and $\{V_n\}$ is the sequence of value regions. Here, the sequence of the continued fraction $\mathbf{K}(-1/-k)$ is the sequence $\{s_n\}$, $s_n(w) = \frac{-1}{-k+w}$ of the Mobius maps, where $a_n = -1$ and $b_n = -k$. $\{S_n\}$ is the sequence of the linear fractional maps produced by $\{s_n\}$. We can defined the sequence $\{S_n\}$ by the recurrence relation of the continued fractions;

$$S_n(w) = s_1 \circ s_2 \circ \cdots \circ s_n(w) = \frac{A_{n-1}w - A_n}{B_{n-1}w + b_n}, \quad n = 1, 2, 3, \dots.$$

By this way, let take the region $K_n = S_n(V_n)$ for the $\{S_n\}$ linear maps. It is clear that for $w \in V_n$, $S_n(w) \in K_n$ holds. Let take $V_0 = \bar{\mathbb{D}} = \{w \in \mathbb{C} : |w| \leq 1\}$, where $\bar{\mathbb{D}}$ is unit disc on the set complex numbers. The images K_n corresponding the value region V_n under the linear fractional maps $\{S_n\}$ are the discs like $K_n = S_n(\bar{\mathbb{D}})$. The images K_n are determined as follows:

$$K_n = S_n(w) = s_1 \circ s_2 \circ \cdots \circ s_n(w), \quad n = 1, 2, 3, \dots.$$

2.1 The discs K_n of the continued fraction $\boldsymbol{K}(-1/-2)$

Let take $k = 2$ on the continued fraction $\mathbf{K}(-1/-k)$. So, the sequence of the continued fraction $\mathbf{K}(-1/-2)$ is $S_n(w) = \frac{-1}{-2+w}$. From recurrence relation and $A_n = (-1)^n n$ on Theorem 1, for $k = 2$, the series of the linear fractional maps $\{S_n\}$, which is produced by $\{s_n\}$ is been written as;

$$S_n(w) = s_1 \circ s_2 \circ \cdots \circ s_n(w) = \frac{(n-1)w - n}{nw - (n+1)}, \quad n = 1, 2, 3, \dots.$$

Here, the sequence of the inverse of the maps S_n is

$$S_n{}^{-1}(w) = \frac{(n+1)w - n}{nw - (n-1)}.$$

Since $K_n = S_n\left(\bar{\mathbb{D}}\right)$, that is $S_n{}^{-1}\left(K_n\right) = \bar{\mathbb{D}}$, for each integer $n \geq 1$, $S_n{}^{-1}(u_n) \in \bar{\mathbb{D}}$ for each $u_n \in K_n$. Then, the following theorem gives the place of region K_n for continued fraction $\mathbf{K}(-1/-k)$:

Theorem 2. *Let $\boldsymbol{K}(-1/-k)$ is a continued fraction and $\{S_n\}$ is the sequence of the linear fractional maps of the $\boldsymbol{K}(-1/-k)$. Let take the $K_n = S_n\left(\bar{\mathbb{D}}\right)$ for region $\bar{\mathbb{D}}$. So, the images K_n are the discs like*

$$K_n = \left\{u_n \in \mathbb{C} : \left|u_n - \frac{2n}{2n+1}\right| \leq \frac{1}{2n+1}\right\}, \quad n = 1, 2, 3, \dots,$$

where the center $C_n = \left(\frac{2n}{2n+1}, 0\right)$ and the radius $R_n = \frac{1}{2n+1}$.

Proof: Proof can be made by mathematical induction method. □

Theorem 3. *Let $\{S_n\}$ is the sequence of the linear fractional maps of the continued fraction* $\mathbf{K}(-1/-2)$ *and* $K_n = S_n\left(\bar{\mathbb{D}}\right)$ *for region* $\bar{\mathbb{D}}$. *Then, the discs* K_n *are the nested discs ;*

$$\bar{\mathbb{D}} \supset K_1 \supset K_2 \supset \cdots \supset K_n.$$

Proof: The continued fraction $\mathbf{K}(-1/-2)$ is a Prinsheim fraction, since $k = 2$. So, for the disc $\bar{\mathbb{D}}$ and the sequence $\{s_n\}$ of the linear fraction maps of the continued fraction $\mathbf{K}(-1/-2)$, we get $s_n\left(\bar{\mathbb{D}}\right) \subset \bar{\mathbb{D}}$. In that case,

$$\begin{aligned} K_n &= S_n\left(\bar{\mathbb{D}}\right) = S_{n-1}\left(s_n\left(\bar{\mathbb{D}}\right)\right) \subseteq S_{n-1}\left(\bar{\mathbb{D}}\right) = K_{n-1}, \\ K_{n-1} &= S_{n-1}\left(\bar{\mathbb{D}}\right) = S_{n-2}\left(s_{n-1}\left(\bar{\mathbb{D}}\right)\right) \subseteq S_{n-2}\left(\bar{\mathbb{D}}\right) = K_{n-2}, \\ &\vdots \\ K_2 &= S_2\left(\bar{\mathbb{D}}\right) = S_1\left(s_2\left(\bar{\mathbb{D}}\right)\right) \subseteq S_1\left(\bar{\mathbb{D}}\right) = K_1 \end{aligned}$$

holds. Then, we write

$$K_n \subset K_{n-1} \subset \cdots \subset K_2 \subset K_1 \subset \bar{\mathbb{D}}.$$

□

To examine the closed discs K_n for $k = 2$ visually, we give the following algorithm on Ptyhon language for first five discs;

```
import matplotlib.pyplot as plt

circle1 = plt.Circle((2/3,0), 1/3, color='black')
circle2 = plt.Circle((4/5,0), 1/5, color='dimgrey')
circle3 = plt.Circle((6/7,0), 1/7, color='slategrey')
circle4 = plt.Circle((8/9,0), 1/9, color='darkgrey')
circle5 = plt.Circle((10/11,0), 1/11, color='lightgrey',clip_on=False)

fig, axes = plt.subplots()

plt.xlim(0,1,)
plt.ylim(-0.5,0.5)

axes.add_artist(circle1)
axes.add_artist(circle2)
axes.add_artist(circle3)
axes.add_artist(circle4)
axes.add_artist(circle5)

fig.savefig('plotcircles.png')
```

Fig. 2

The discs which are drawn with this algorithm, are as the Figure 3. Here, it is seen that, the discs K_n are tangent at point 1. For $k = 2$, the

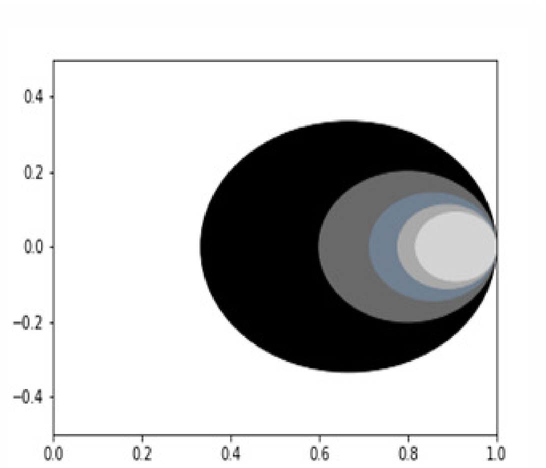

Fig. 3

following corollary is about the convergence of the discs;

Corollary 4. *There is a boundary point condition for the sequence $\{K_n\}$ which is composed by $K_n = S_n(\bar{\mathbb{D}})$ for the continued fraction $\mathbf{K}(-1/-2)$ and the closed disc $\bar{\mathbb{D}}$. The sequence converges to $f = 1$.*

Proof: Since the discs K_n are the subsets of the closed discs $\bar{\mathbb{D}}$, they are closed discs. Then, the discs K_n are formed by the sequences, for the radius $\{R_n\} = \left\{0, \frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2n+1}\right\}$ and for the center $\{C_n\} = \left\{1, \frac{2}{3}, \frac{4}{5}, \ldots, \frac{2n}{2n+1}\right\}$. So, the following limits are written;

$$\lim R_n = \lim_{n\to\infty} \frac{1}{2n+1} = 0 \quad \textit{and} \quad \lim C_n = \lim_{n\to\infty} \frac{2n}{2n+1} = 1.$$

Because of $\lim R_n = 0$, there is boundary point condition and it converges to $\lim C_n = 1$.

□

2.2 The discs K_n of the continued fraction $\mathbf{K}(-1/-3)$

For $k = 3$ the sequence of the maps of the continued fraction $\mathbf{K}(-1/-3)$ is $s_n(w) = \frac{-1}{-3+w}$. $\{S_n\}$, which is produced by $\{s_n\}$ of the disc $\bar{\mathbb{D}}$ is the sequence of linear fractional maps of the linear fractional maps. When we use the equation on Corollary 2, the elements of that sequence are represented by Fibonacci numbers;

$$S_n(w) = s_1 \circ s_2 \circ \cdots \circ s_n(w) = \frac{-F_{2n-2}w + F_{2n}}{-F_{2n}w + F_{2n+2}}, \quad n = 1, 2, 3, \ldots .$$

Here, the sequence of the inverse of the maps S_n is

$$S_n{}^{-1}(w) = \frac{F_{2n+2}w - F_{2n}}{F_{2n}w - F_{2n-2}}, \quad n = 1, 2, 3, \ldots .$$

Theorem 4. *Let $\mathbf{K}(-1/-3)$ is a continued fraction and $\{S_n\}$ is the sequence of the linear fractional maps of the $\mathbf{K}(-1/-3)$. Let take the $K_n = S_n\left(\bar{\mathbb{D}}\right)$ for region $\bar{\mathbb{D}}$. So, the images K_n are the discs like*

$$K_n = \left\{v_n \in \mathbb{C} : \left|v_n - \frac{F_{4n}}{F_{4n+2}}\right| \le \frac{1}{F_{4n+2}}\right\}, \quad n \ge 1,$$

where the center $C_n = \left(\frac{F_{4n}}{F_{4n+2}}, 0\right)$ and the radius $R_n = \frac{1}{F_{4n+2}}$.

Proof: For the proof, we use mathematical induction method. For $n = 1$ and $v_1 \in K_1$, the inverse map S_1^{-1} is

$$S_1{}^{-1}(v_1) = \frac{F_4 v_1 - F_2}{F_2 v_1 - F_0},$$

where $K_1 = S_1(\bar{\mathbb{D}})$. Since $S_1{}^{-1}(v_1) \in \bar{\mathbb{D}}$, the place of disc K_1 is obtained by solving the expression $\left|\frac{F_4 v_1 - F_2}{F_2 v_1 - F_0}\right| \le 1$. So, for $v_1 = x_1 + iy_1$,

$$\left|\frac{F_4 x_1 + iF_4 y_1 - F_2}{F_2 x_1 + iF_2 y_1 - F_0}\right| \le 1$$

holds. When we solve the inequality, we get

$$\left(F_4{}^2 - F_2{}^2\right) x_1{}^2 - 2x_1 F_2 \left(f_4 - F_0\right) + \left(F_4{}^2 - F_2{}^2\right) y_1{}^2 + F_2{}^2 - F_0{}^2 \le 0.$$

From the identities $F_{2n} = F_n L_n$, $F_{n-1} + F_{n+1} = L_n$ and $F_{n+2} - F_{n-2} = L_n$, we write

$$x_1{}^2 - 2\frac{F_4}{F_6}x_1 + y_1{}^2 + \frac{F_2}{F_6} \le 0.$$

Therefore, the place of disc K_1 is given as follows

$$\left(x_1 - \frac{F_4}{F_6}\right)^2 + y_1{}^2 \le \left(\frac{1}{F_6}\right)^2.$$

So, this gives that K_1 is a closed disc with center $\left(\frac{F_4}{F_6}, 0\right)$ and radius $\frac{1}{F_6}$. Now, let assume the statement is true for K_n. We should show that the statement is true for K_{n+1}. We can write

$$S_{n+1}(w) = S_n\left(s_{n+1}(w)\right) = \frac{-F_{2n}w + F_{2n+2}}{-F_{2n+2}w + F_{2n+4}} \quad \textit{and} \quad S_{n+1}{}^{-1}(w) = \frac{F_{2n+4}w - F_{2n+2}}{F_{2n+2}w - F_{2n}}.$$

Since, for all $v_{n+1} \in K_{n+1}$, ${S_{n+1}}^{-1}(v_{n+1}) \in \bar{\mathbb{D}}$ is written, the place of disc K_1 is obtained by solving the expression $\left|\frac{F_{2n+4}v_{n+1}-F_{2n+2}}{F_{2n+2}v_{n+1}-F_{2n}}\right| \leq 1$. So, for $v_{n+1} = x_{n+1} + iy_{n+1}$,

$$\left|\frac{F_{2n+4}x_{n+1} + iF_{2n+4}y_{n+1} - F_{2n+2}}{F_{2n+2}x_{n+1} + iF_{2n+2}y_{n+1} - F_{2n}}\right| \leq 1$$

holds. When we solve the inequality and use the following identities

$$F_{2n+4}^2 - F_{2n+2}^2 = F_{4n+6}, \;\; F_{2n+2}\left(F_{2n+4} - F_{2n}\right) = F_{4n+4}, \;\; F_{2n+2}^2 - F_{2n}^2 = F_{4n+2},$$

we get

$$x_{n+1}^2 - 2\frac{F_{4n+4}}{F_{4n+6}}x_{n+1} + y_{n+1}^2 + \frac{F_{4n+2}}{F_{4n+6}} \leq 0.$$

Therefore, the place of disc K_{n+1} is given as follows

$$\left(x_{n+1} - \frac{F_{4(n+1)}}{F_{4(n+1)+2}}\right)^2 + y_{n+1}^2 \leq \left(\frac{1}{F_{4(n+1)+2}}\right)^2.$$

So, this gives that K_{n+1} is a closed disc with center $\left(\frac{1}{F_{4(n+1)+2}}, 0\right)$ and radius $\frac{1}{F_{4(n+1)+2}}$. Then the proof is done. □

Theorem 5. *Let $\{S_n\}$ is the sequence of the linear fractional maps of the continued fraction $\boldsymbol{K}(-1/-3)$ and $K_n = S_n\left(\bar{\mathbb{D}}\right)$ for region $\bar{\mathbb{D}}$. Then, the discs K_n are the nested discs ;*

$$\bar{\mathbb{D}} \supset K_1 \supset K_2 \supset \cdots \supset K_n.$$

Proof: The continued fraction $\mathbf{K}(-1/-3)$ is a Prinsheim fraction, since $k = 3$. So, for the disc $\bar{\mathbb{D}}$ and the sequence $\{s_n\}$ of the linear fraction maps of the continued fraction $\mathbf{K}(-1/-3)$, we get $s_n\left(\bar{\mathbb{D}}\right) \subset \bar{\mathbb{D}}$. In that case,

$$\begin{aligned}
K_n &= S_n\left(\bar{\mathbb{D}}\right) = S_{n-1}\left(s_n\left(\bar{\mathbb{D}}\right)\right) \subseteq S_{n-1}\left(\bar{\mathbb{D}}\right) = K_{n-1}, \\
K_{n-1} &= S_{n-1}\left(\bar{\mathbb{D}}\right) = S_{n-2}\left(s_{n-1}\left(\bar{\mathbb{D}}\right)\right) \subseteq S_{n-2}\left(\bar{\mathbb{D}}\right) = K_{n-2}, \\
&\vdots \\
K_2 &= S_2\left(\bar{\mathbb{D}}\right) = S_1\left(s_2\left(\bar{\mathbb{D}}\right)\right) \subseteq S_1\left(\bar{\mathbb{D}}\right) = K_1
\end{aligned}$$

holds. Then, we write

$$K_n \subset K_{n-1} \subset \cdots \subset K_2 \subset K_1 \subset \bar{\mathbb{D}}.$$

□

To examine the closed discs K_n for $k = 3$ visually, we give the following algorithm on Ptyhon language for first five discs;

```
import matplotlib.pyplot as plt

circle1 = plt.Circle((3/8,0), 1/8, color='lightgrey')
circle2 = plt.Circle((21/55,0), 1/55, color='grey')
circle3 = plt.Circle((144/377,0), 1/377, color='black',clip_on=False)

fig, axes = plt.subplots()

plt.xlim(0.2,0.5)
plt.ylim(-0.15,0.15)

axes.add_artist(circle1)
axes.add_artist(circle2)
axes.add_artist(circle3)


fig.savefig('plotcircles.png')
```

Fig. 4

The discs which are drawn with this algorithm, are as the Figure 4. Here, it is seen that, the discs K_n are tangent at point 1. For $k = 2$, the following corollary is about the convergence of the discs;

Corollary 5. *There is a boundary point condition for the sequence $\{K_n\}$ which is composed by $K_n = S_n(\bar{\mathbb{D}})$ for the continued fraction $\boldsymbol{K}(-1/-3)$ and the closed disc $\bar{\mathbb{D}}$. The sequence converges to $f = \frac{3-\sqrt{5}}{2}$.*

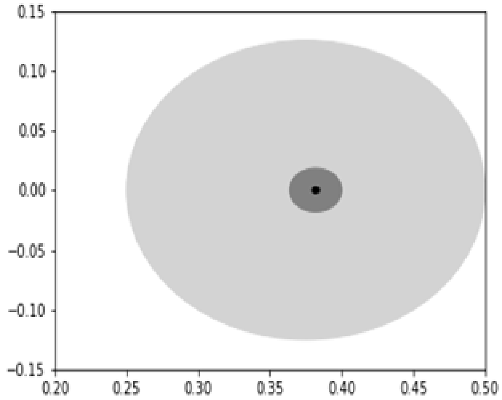

Fig. 5

Proof: Since the discs K_n are the subsets of the closed discs $\bar{\mathbb{D}}$, they are closed discs. Then, the discs K_n are formed by the sequences, for the radius $\{R_n\} = \left\{0, \frac{1}{F_6}, \frac{1}{F_{10}}, \dots, \frac{1}{F_{4n+2}}\right\}$ and for the center $\{C_n\} = \left\{1, \frac{F_4}{F_6}, \frac{F_8}{F_{10}}, \dots, \frac{F_{4n}}{F_{4n+2}}\right\}$. So, the following limits are written;

$$\lim R_n = \lim_{n\to\infty} \frac{1}{F_{4n+2}} = 0 \quad \textit{and} \quad \lim C_n = \lim_{n\to\infty} \frac{F_{4n}}{F_{4n+2}} = 1 - \frac{1}{\alpha} = \frac{3-\sqrt{5}}{2}.$$

Because of $\lim R_n = 0$, there is boundary point condition and it converges to $\lim C_n = \frac{3-\sqrt{5}}{2}$.

□

3 References

1 A.H. Değer, M. Beşenk, B.O. Güler, *On Suborbital Graphs And Related Continued Fractions*, Applied Mathematics And Computation, **218** (2011), 746-750.
2 A.H. Değer, *Vertices Of Paths Of Minimal Lengths On Suborbital Graphs*, Filomat, **31(4)** (2017), 913-923.
3 Ü. Akbaba, A.H. Değer, T. Tuylu, *On Some Connections Between Suborbital Graphs and Special Sequences* , Turkish Journal of Mathematics and Computer Science, **10** (2018), 134-143.
4 W.B. Jones, W.J. Thron, *Continued Fractions, Encyclopedia of Mathematics and its Applications*, Addison-Wesley, 1980.
5 A. F. Beardon, *The Geometry of Pringsheim's Continued Fractions*, Geometriae Dedicata, **84** (2001), 125-134.

